PHYS 512

Assignment – 04 $\langle \boldsymbol{Quantum} | \boldsymbol{Computing} \rangle$

Due Tuesday, Oct 014 – 11:00 pm

Submit Question # 1, 3, 6, 8, 9 and 11 ONLY (You are strongly encouraged to solve all of them)

Q.1 Spectral decomposition of an Operator

Using spectral decomposition theorem, write down the following operator

$$\hat{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

In Standard notation $\hat{A} = \sum_{i=1}^{n} a_i |u_i\rangle \langle u_i|$ where a_i are the Eigen values and u_i are eigen states of the operator A.

(Hint: First find the eigen values and eigen operators of the operator A)

Q.2 Commutation relation

(a) Show that

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$

Q.3 Bloch Sphere

(a) Show that

$$\hat{n}.\vec{\sigma} = \begin{pmatrix} cos\theta & e^{-i\varphi} sin\theta \\ e^{i\varphi} sin\theta & -cos\theta \end{pmatrix}$$

Where $\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$ and $\hat{n} = sin\theta cos\phi\hat{x} + sin\theta sin\phi\hat{y} + cos\theta\hat{z}$

(b) Find the value of θ and φ for which (using Bloch Sphere)

(i) $\hat{n}.\vec{\sigma} = \sigma_z$

(ii) $\hat{n}.\vec{\sigma} = \sigma_y$

(iii) $\hat{n}.\vec{\sigma} = \sigma_x$

(c) We know from class lecture, any possible state $|\psi\rangle$ of a qubit can be represented by a point the surface of Bloch sphere, where

$$|\psi\rangle = \begin{pmatrix} cos\theta/2 \\ e^{i\varphi} sin\theta/2 \end{pmatrix}$$

Find the value of θ and φ for which the above state $|\psi\rangle$ becomes a

(i) $|+\rangle$

(ii) $|-\rangle$

(d)

(i) Show that computational basis $\{|0\rangle \; and \; |1\rangle\}$ are eigen states of σ_z

(ii) Show that computational basis $\{|0\rangle \; and \; |1\rangle\}$ are NOT eigen states of σ_x

(iii) Show that basis $|+\rangle \; and \; |-\rangle$ are eigen states of σ_x

(Note for above don't assume they are eigen state, explicitly show by matrix/ ket (/bra) multiplication)

Q.4 Gram-Schmidt Orthogonalization

Use the gram-Schmidt process to construct an orthonormal basis set from

$$|v_1\rangle = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad |v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad |v_3\rangle = \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix}$$

(Hint: Following is the procedure to produce, for more details Book-2)

GRAM-SCHMIDT ORTHOGONALIZATION

An orthonormal basis can be produced from an *arbitrary* basis by application of the *Gram-Schmidt orthogonalization* process. Let $\{|v_1\rangle, |v_2\rangle, \ldots\}|v_n\rangle$, be a basis for an inner product space V. The Gram-Schmidt process constructs an orthogonal basis $|w_i\rangle$ as follows:

$$|w_1\rangle = |v_1\rangle$$

$$|w_2\rangle = |v_2\rangle - \frac{\langle w_1|v_2\rangle}{\langle w_1|w_1\rangle}|w_1\rangle$$

$$\vdots$$

$$|w_n\rangle = |v_n\rangle - \frac{\langle w_1|v_n\rangle}{\langle w_1|w_1\rangle}|w_1\rangle - \frac{\langle w_2|v_n\rangle}{\langle w_2|w_2\rangle}|w_2\rangle - \cdots - \frac{\langle w_{n-1}|v_n\rangle}{\langle w_{n-1}|w_{n-1}\rangle}|w_{n-1}\rangle$$

To form an orthonormal set using the Gram-Schmidt procedure, divide each vector by its norm. For example, the normalized vector we can use to construct $|w_2\rangle$ is

$$|w_2\rangle = \frac{|v_2\rangle - \langle w_1|v_2\rangle|w_1\rangle}{\||v_2\rangle - \langle w_1|v_2\rangle|w_1\rangle\|}$$

Many readers might find this a bit abstract, so let's illustrate with a concrete example.

Q.5 Pauli Matrix and Rotations

Rotation along x-axis by angle θ can expressed as

$$R_x(\theta) = e^{-i\frac{\theta}{2}\hat{X}}$$

Where $\hat{X}$ is a pauli-X operator

a) Using Taylor series expansion show that

$$e^{-i\frac{\theta}{2}\hat{X}} = \hat{I}\,cos(\theta/2) - i\hat{X}sin(\theta/2)$$

b) Also show that

$$e^{-i\frac{\theta}{2}\hat{Z}} = \hat{I}\,cos(\theta/2) - i\hat{Z}sin(\theta/2)$$

c) Further show that rotation along z-axis can be expressed as

$$e^{-i\frac{\theta}{2}\hat{Z}} = \begin{pmatrix} e^{\frac{-i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}$$

Q.6 General Unitary

Any unitary operator can be expressed as

$$U = e^{i\alpha}\left[R_z(\beta)R_y(\gamma)R_z(\delta)\right]$$

Show that above unitary operator can be expressed as

$$U = e^{i\alpha}\begin{pmatrix} e^{\frac{-i\beta}{2}-\frac{i\delta}{2}}cos(\gamma/2) & e^{\frac{-i\beta}{2}+\frac{i\delta}{2}}sin(\gamma/2) \\ e^{\frac{i\beta}{2}-\frac{i\delta}{2}}sin(\gamma/2) & e^{\frac{i\beta}{2}+\frac{i\delta}{2}}cos(\gamma/2) \end{pmatrix}$$

Q.7 *Tensor Product*

Find the tensor product of

(i) Find the tensor product of (i.e. $|\psi\rangle \otimes |\varphi\rangle$)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad and \quad |\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$$

(ii) Find the tensor product of

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \quad and \quad \sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

(iii) Find

$$\sigma_x \otimes \sigma_y |\psi\rangle = ?$$

Where

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Q.8 [1+4+2+4] ***Quantum Gates* – CNOT**

Note modular two addition is defined as:

$$0 \oplus 0 = 0; \quad 0 \oplus 1 = 1; \quad 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0$$

(a) If

$$\text{CNOT } |a\ b\rangle = |a\ (a \oplus b)\rangle$$

(Where on right hand side ‘a’ is first qubit and $(a \oplus b)$is second qubit)

Using above definition of CNOT-gate complete the following

$$\text{CNOT } |00\rangle = ?$$
$$\text{CNOT } |01\rangle = ?$$
$$\text{CNOT } |10\rangle = ?$$
$$\text{CNOT } |11\rangle =$$

(b)

(i) Find

$$(\text{CNOT})_3 (\text{CNOT})_2 (\text{CNOT})_1 |10\rangle = \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

Where CNOT_1 and CNOT_3 are **usual** CNOT-gates, but for CNOT_2, **control bit is second qubit**

(ii) Draw the quantum circuit for equation (1)

(c) CNOT gate expressed in terms of Pauli gates

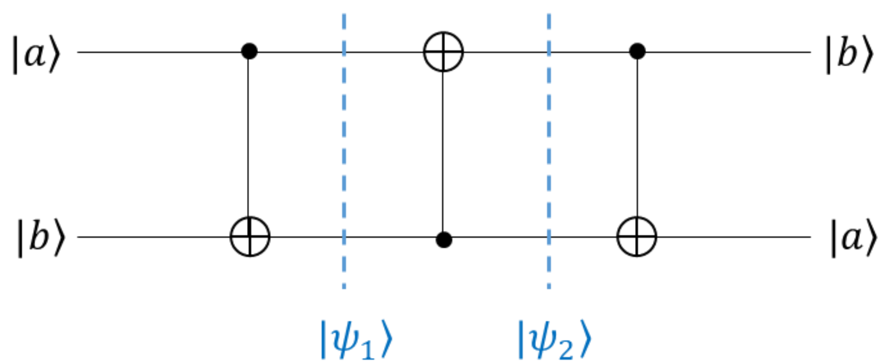
$$= \frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2}\boldsymbol{\sigma}_Z \otimes \mathbb{1} + \frac{1}{2}\mathbb{1} \otimes \boldsymbol{\sigma}_X - \frac{1}{2}\boldsymbol{\sigma}_Z \otimes \boldsymbol{\sigma}_X.$$

Using above expression show that the matrix representation of CNOT is

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Q.9 (Multiple CNOT)

Show that following combinations of CNOT gates just flip the inputs.
(Do not write the final answer directly**,** Find $|\psi_1\rangle$ and $|\psi_2\rangle$ as well)



(Hint: use modular two definition of CNOT i.e. CNOT$|ab\rangle = |a\ (a \oplus b)\rangle$. Find $|\psi_1\rangle$ and $|\psi_1\rangle$

Q.10 *Quantum Gate*s – Toffoli Gate

Complete the following truth table for Toffoli gate

(Remember; $|c'\rangle = |c \oplus ab\rangle$

Inputs			Outputs		
$\|a\rangle$	$\|b\rangle$	$\|c\rangle$	$\|a'\rangle$	$\|b'\rangle$	$\|c'\rangle$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Q.11 [4+3+4Quantum Circuits

(a) Draw Quantum circuit for the following equation

$$(\mathrm{CNOT} \otimes \mathrm{I})(\mathrm{X} \otimes \mathrm{H} \otimes \mathrm{Z})|110\rangle$$

(b) Find the state $|\psi_1\rangle$

$$|\psi_1\rangle = (\mathrm{X} \otimes \mathrm{H} \otimes \mathrm{Z})|110\rangle$$

(c) Find the state $|\psi_2\rangle$

$$|\psi_2\rangle = (\mathrm{CNOT} \otimes \mathrm{I})(\mathrm{X} \otimes \mathrm{H} \otimes \mathrm{Z})|110\rangle$$