Q.1 41 Points Basic Concepts

1. Which one of the following can be a possible eigenvalues of a Hermitian operator (circle all possible answer(s)) [1]

Solution:
Eigenvalues of a Hermitian operator must be real.
$(2+3i)$ – complex, not an eigenvalue
$4i$ – imaginary, not an eigenvalue
2.5 – real ✓
$2\sqrt{3}$ – real ✓

2. All Pauli matrices are Hermitian and unitary (Y/N) [1]

Solution:
Yes

3. Express the Hadamard operator (gate) in ket/bra notation (no need to derive) [1]

Solution:

$$H = |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|$$

4. Consider a point P on a Bloch sphere as shown in the figure, where $\theta = 60^\circ$ and state vector $|\psi\rangle$ is right above positive y-axis (in yz-plane). Find the expression for state vector $|\psi\rangle$ in ket/bra (Dirac) notation and in matrix form. [2]

Solution:

$$\varphi = 90^\circ$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$|\psi\rangle = \cos(30^\circ)|0\rangle + e^{i\frac{\pi}{2}}\sin(30^\circ)|1\rangle$$

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

Matrix form:

$$|\psi\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{i}{2} \end{pmatrix}$$

5. Show that the operator $|0\rangle\langle 1|$ is equivalent $\frac{1}{2}(\sigma_x + i\sigma_y)$ **[2]**

Solution:

$$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\begin{aligned}\frac{1}{2}(\sigma_x + i\sigma_y) &= \frac{1}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 1| - |1\rangle\langle 0|\right)\\ &= \frac{1}{2}(2|0\rangle\langle 1|) = |0\rangle\langle 1| \qquad \square\end{aligned}$$

6. $YZX|1\rangle = ?$ **[2]**

Solution:

$$\begin{aligned}YZX|1\rangle &= YZ|0\rangle = Y|0\rangle\\ &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ i\end{pmatrix} = i|1\rangle\end{aligned}$$

Also:

$$-iXYZ = I \Rightarrow XYZ = iI$$

Therefore:

$$YZX|1\rangle = i|1\rangle$$

7. Show that $Z = HXH$ **[2]**

Where Z (σ_z) and X (σ_x) are Pauli gates (/operators) and H is Hadamard gate

Solution:

$$\begin{aligned}HXH &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = Z \qquad \square\end{aligned}$$

8. Show that $[A,B]^\dagger = [B^\dagger, A^\dagger]$ **[2]**

Solution:

$$\begin{aligned}(AB - BA)^\dagger &= B^\dagger A^\dagger - A^\dagger B^\dagger\\ (AB - BA)^\dagger &= (AB)^\dagger - (BA)^\dagger\\ &= B^\dagger A^\dagger - A^\dagger B^\dagger\\ &= [B^\dagger, A^\dagger] \qquad \square\end{aligned}$$

9. Consider the following phase gate operator P [2]

$$P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

Show that when P act on *general qubit state* $|\psi\rangle$, it changes the phase of qubit by $\pi/2$.

Solution:

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$P|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\frac{\pi}{2}}e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\varphi+\frac{\pi}{2})}\sin\frac{\theta}{2} \end{pmatrix}$$

Therefore (New Phase):

$$\tilde{\varphi} = \varphi + \frac{\pi}{2}$$

□

10. Consider following two quantum states [2]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Show that above two state are orthogonal?

Solution:

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}}(1\ 0\ 0\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}[0+0+0+0] = 0 \qquad \square$$

11. $\hat{X}, \hat{Y}, \&\ \hat{Z}$ are Pauli operators. Express the following expression in matrix form. [3]

$$\hat{X} \otimes \hat{X} + \hat{Y} \otimes \hat{Y} + \hat{Z} \otimes \hat{Z}$$

Solution:

$$\hat{X} \otimes \hat{X} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{Z} \otimes \hat{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{Y} \otimes \hat{Y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \square$$

12. Consider following three qubit state [2]

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|101\rangle + |011\rangle]$$

If $\hat{P} = |1\rangle\langle 1|$ is a projection operator (remember $\hat{I}$ is identity operator). Find

$$\hat{P} \otimes \hat{I} \otimes \hat{I}|\psi\rangle = ?$$

Solution:
P is a projection of the 1st qubit in $|1\rangle$ state, hence only $|101\rangle$ remains:

$$|\psi\rangle = |101\rangle$$

$$\hat{P} \otimes \hat{I} \otimes \hat{I}|0xx\rangle = 0$$

$$\hat{P} \otimes \hat{I} \otimes \hat{I}|1xx\rangle = |1xx\rangle \qquad \square$$

13. Find the final state after applying the gates. Is final state entangled? [2]

$$(\text{CNOT})(H \otimes I)|00\rangle$$

Solution:

$$H \otimes I|00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

$$\begin{aligned}\text{CNOT}\left[\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right] &= \frac{\text{CNOT}|00\rangle + \text{CNOT}|10\rangle}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\end{aligned}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

It is entangled and is a Bell state. □

14. Find the eigen values of the given operator [3]

$$\hat{A} = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

Solution:

$$|\hat{A} - \lambda \hat{I}| = 0$$

$$\begin{vmatrix} -1-\lambda & 0 & 2 \\ 2 & -\lambda & -1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = -1 \cdot \lambda \begin{vmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} - 0 + 2 \begin{vmatrix} 2 & -\lambda \\ 2 & 1 \end{vmatrix} = 0$$

$$(-1-\lambda)(-\lambda(2-\lambda)+1) + 2(2+2\lambda) = 0$$

$$-1(1+\lambda)[(\lambda^2 - 2\lambda + 1)] = 0$$

$$= (1+\lambda)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda - 3)(\lambda + 1)^2 = 0$$

$$\lambda = -1, -1, 3 \qquad \square$$

15. A projection $\hat{P}$ operator is defined as [2]

$$\hat{P} = |i\rangle\langle i|$$

Show that

$$\hat{P}^2 = P$$

Solution:

$$\begin{aligned}\hat{P}^2 &= (|i\rangle\langle i|)(|i\rangle\langle i|) \\ &= |i\rangle\langle i|i\rangle\langle i|\end{aligned}$$

Assuming $|i\rangle$ is normalized:

$$\langle i|i\rangle = 1$$

$$\therefore \hat{P}^2 = |i\rangle\langle i| = \hat{P} \qquad \square$$

16. Show that (where X, Y, and Z are Pauli operators) [3]

$$[X, Y] = 2iZ$$

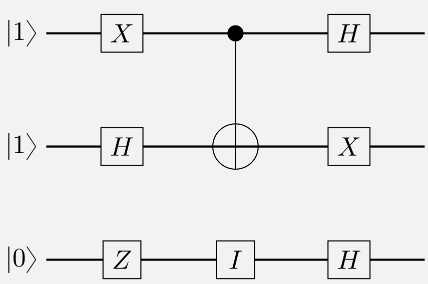
Solution:

$$\begin{aligned}[X, Y] &= XY - YX \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2iZ \qquad \square\end{aligned}$$

17. Draw the circuit for the following gate operation [3]

$$(H \otimes X \otimes H)(\text{CNOT} \otimes I)(X \otimes H \otimes Z)|110\rangle$$

Solution:
Quantum circuit (reading right to left):



$\square$

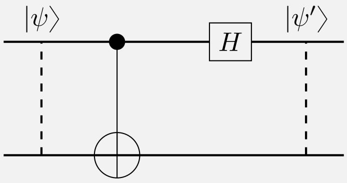
18. If **[4]**

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

(a) Draw quantum circuit for the following operation on $|\psi\rangle$

$$(H \otimes I)(\text{CNOT})|\psi\rangle$$

Solution:



(b) Find the final state $|\psi'\rangle$

$$(H \otimes I)(\text{CNOT})|\psi\rangle = |\psi'\rangle$$

Solution:

$$\begin{aligned} \text{CNOT}\left[\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right] &= \frac{\text{CNOT}|01\rangle - \text{CNOT}|10\rangle}{\sqrt{2}} \\ &= \frac{|01\rangle - |11\rangle}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} (H \otimes I)\left[\frac{|01\rangle - |11\rangle}{\sqrt{2}}\right] &= \frac{1}{\sqrt{2}}[H|0\rangle \otimes |1\rangle - H|1\rangle \otimes |1\rangle] \\ &= \frac{1}{\sqrt{2}}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |1\rangle - \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |1\rangle\right] \\ &= \frac{1}{2}[|01\rangle + |11\rangle - |01\rangle + |11\rangle] \\ &= \frac{1}{2}[2|11\rangle] = |11\rangle \end{aligned}$$

Final state:

$$|\psi'\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

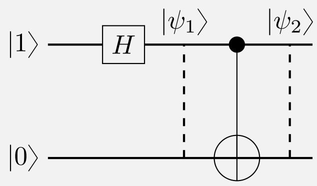
19. For given input find the state $|\psi_1\rangle$ and $|\psi_2\rangle$. Is the state $|\psi_2\rangle$ an entangled state? **[2]**

Solution:

$$\psi_1 = H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\begin{aligned}\psi_2 = \text{CNOT}\psi_1 &= \text{CNOT}\left[\frac{|00\rangle - |10\rangle}{\sqrt{2}}\right] \\ &= \frac{\text{CNOT}|00\rangle - \text{CNOT}|10\rangle}{\sqrt{2}} \\ &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}\end{aligned}$$

$|\psi_2\rangle$ is entangled and is a bell state. □



Q.2 6 Points Measurements and renormalization

Consider the following state

$$|\psi\rangle = \frac{2}{3}|101\rangle + \frac{1}{3}|011\rangle + \frac{2}{3}|111\rangle$$

Suppose that *first qubit* of the state $|\psi\rangle$ is measured in the computational basis.
(a) What is the probability of obtaining $|1\rangle$ [4]
(b) If measurement outcome of first qubit is $|1\rangle$ what is (are) possible state(s) of second qubit?
(c) After the measuring the first qubit in $|1\rangle$, write down the new (normalized) state (and call it $|\varphi'\rangle$)

Solution:
a)

$$\left|\frac{2}{3}\right|^2 + \left|\frac{2}{3}\right|^2 = \frac{4}{9} + \frac{4}{9} = \frac{8}{9} \approx 89\%$$

b)

$$|\Psi\rangle = \frac{|101\rangle + |111\rangle}{2}$$

Hence for Q_2 could be in $|0\rangle$ or $|1\rangle$
c)

$$|\varphi'\rangle = \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

Now suppose now the second qubit of $|\varphi'\rangle$ is measured in the computational basis. []
(d) What is the probability of obtaining in state $|0\rangle$?
(e) (b) after the measurement of second qubit in state $|0\rangle$, what is the resulting (normalized) state $|\varphi''\rangle$?

Solution:
d)

$$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \approx 50\%$$

e)

$$|\varphi''\rangle = |101\rangle$$

Q.3 8 Points Normalization, possible outcomes and expectation value

An X-ray fluorescence spectrometer *analyzes the energy* of an X-ray photon. Energy H is an *observable* and the corresponding *Hermitian operator* is $\hat{H}$.

XRF Spectrometer

The incoming state is

$$|\psi\rangle = \sqrt{2}|1\rangle + \sqrt{3}|2\rangle + |3\rangle + |4\rangle$$

where $|1\rangle, |2\rangle, |3\rangle, and |4\rangle$ are the nondegenerate eigenstates of $\hat{H}$, such that

$$\hat{H}|n\rangle = n^2 \varepsilon_o |n\rangle$$

and ε_o is a real constant with dimensions of energy.

(a) Normalize the input state$|\psi\rangle$ [1]

Solution:

$$\langle\psi|\psi\rangle = \sqrt{2+3+1+1} = \sqrt{7}$$

$$|\psi\rangle = \frac{1}{\sqrt{7}}\left[\sqrt{2}|1\rangle + \sqrt{3}|2\rangle + |3\rangle + |4\rangle\right]$$

(b) What is the probability that the XRF measurement out outcome is $|3\rangle$ [1]

Solution:

$$\left|\frac{1}{\sqrt{7}}\right|^2 = \frac{1}{7} \approx 14\%$$

(c) Express the operator $\hat{H}$ in matrix form in $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ basis. [2]

Solution:
Since $\hat{H}$ is diagonal in its eigenstate basis:

$$\hat{H} = \begin{pmatrix} \varepsilon_o & 0 & 0 & 0 \\ 0 & 4\varepsilon_o & 0 & 0 \\ 0 & 0 & 9\varepsilon_o & 0 \\ 0 & 0 & 0 & 16\varepsilon_o \end{pmatrix}$$

Since H is diagonal in its eigenstate basis and

$$\hat{H}|n\rangle = n^2 \varepsilon_o |n\rangle$$

it is evident that it is diagonal. □

(d) Find the expectation value of $\langle\hat{H}\rangle$ (average energy measured) for given state $|\psi\rangle$ [4]

Solution:

$$\begin{aligned}\langle\hat{H}\rangle &= \langle\psi|\hat{H}|\psi\rangle \\ &= \frac{1}{7}\left[\sqrt{2}\langle 1| + \sqrt{3}\langle 2| + \langle 3| + \langle 4|\right]\hat{H}\left[\sqrt{2}|1\rangle + \sqrt{3}|2\rangle + |3\rangle + |4\rangle\right]\end{aligned}$$

$$= \frac{1}{7}\left[2\varepsilon_o + 12\varepsilon_o + 9\varepsilon_o + 16\varepsilon_o\right] = \frac{39}{7}\varepsilon_o$$

Q.4 12 Points Time evolution of quantum state

a) Express the following rotation operator $R_i(\theta)$ (rotation along i-axis) in matrix form [4]

$$R_x(\theta) = e^{-i\frac{\theta}{2}\hat{X}}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\hat{Y}}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\hat{Z}}$$

Solution:
For $R_x(\theta)$:

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{X}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} - \begin{pmatrix} 0 & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

For $R_y(\theta)$:

$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{Y}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} - \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

For $R_z(\theta)$:

$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{Z}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} - \begin{pmatrix} i\sin\frac{\theta}{2} & 0 \\ 0 & -i\sin\frac{\theta}{2} \end{pmatrix}$$
$$= \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

Using:

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} = e^{-i\frac{\theta}{2}}$$

(b) Initial state of the qubit is $|0\rangle$
(i) If you **rotate this initial state by** 90° **about the y-axis**, what is the new state? [1]

Solution:

$$|\psi\rangle = |+\rangle$$

(Rotating $|0\rangle$ by 90° about the y-axis on the Bloch sphere brings it to the $|+\rangle$ state)

(ii) Verify your answer by applying rotation operator $R_y(\theta)$ (with suitable θ) on state $|0\rangle$ [3]

Solution:

$$R_y^{90^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(iii) The rotation you used in part (ii) is a Hadamard gate. (Y/N) [1]

Solution:
No

(c) Evaluate the following and express final answer in $\{|+\rangle, |-\rangle\}$ basis. [3]

$$R_z(180^\circ)|+\rangle$$

Solution:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\pi}{2}} & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\pi}{2}} \\ e^{i\frac{\pi}{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ i \end{pmatrix} = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -i|-\rangle$$

Additional Information

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$[A, B] = AB - BA$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle; \qquad |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

$$\hat{n} = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta);$$

$$\langle \hat{A} \rangle = \langle \psi|\hat{A}|\psi\rangle; \qquad S.D = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

$$|\psi\rangle = \begin{pmatrix} cos\theta/2 \\ e^{i\varphi} sin\theta/2 \end{pmatrix}$$

$$e^{-i\theta} = cos\theta - isin\theta$$

$$R_z(\theta) = cos\left(\frac{\theta}{2}\right)\hat{I} - isin\left(\frac{\theta}{2}\right)\hat{Z}$$

$$R_y(\theta) = cos\left(\frac{\theta}{2}\right)\hat{I} - isin\left(\frac{\theta}{2}\right)\hat{Y}$$

$$R_x(\theta) = cos\left(\frac{\theta}{2}\right)\hat{I} - isin\left(\frac{\theta}{2}\right)\hat{X}$$