Density Operator/Matrix

Ch -5 / David McMahon

Important Properties of the Trace

The trace has some important properties that are good to know. These include the following:

- The trace is *cyclic*, meaning that $Tr(ABC) = Tr(CAB) = Tr(BCA)$.
- The trace of an outer product is the inner product $Tr(|\phi\rangle\langle\psi|) = \langle\phi|\phi\rangle$.
- By extension of the above it follows that $Tr(A|\psi\rangle\langle\phi|) = \langle\phi|A|\psi\rangle$.
- The trace is *basis independent*. Let $|u_i\rangle$ and $|v_i\rangle$ be two bases for some Hilbert space. Then $Tr(A) = \sum\langle u_i|A|u_i\rangle = \sum\langle v_i|A|v_i\rangle$.
- The trace of an operator is equal to the sum of its eigenvalues. If the eigenvalues of A are labeled by λ_i, then $Tr(A) = \sum_{i=1}^{n} \lambda_i$.
- The trace is linear, meaning that $Tr(\alpha A) = \alpha Tr(A)$, $Tr(A+B) = Tr(A) + Tr(B)$.

Density Operator

Pure State

$$\rho = |\psi\rangle\langle\psi|$$

Derivation on page 85 – NOT required

Mixed State

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho = p_1\, |\psi_1\rangle\langle\psi_1| + p_2\, |\psi_2\rangle\langle\psi_2| + p_3\, |\psi_3\rangle\langle\psi_3|$$

Expectation value using density operator

$$\langle A\rangle = Tr(\rho A)$$

Trace of density operator of a pure state

$$Tr(\rho) = \sum_{j=1}^{n} \langle u_j|\rho|u_j\rangle$$

Trace is sum of diagonal elements

$$Tr(\rho) = \sum_{j=1}^{n} \langle u_j|\psi\rangle\langle\psi|u_j\rangle$$

j^{th} component of $|\psi\rangle$

$$Tr(\rho) = \sum_{j=1}^{n} C_j C_j^* = \sum_j |C_j|^2 = 1$$

$Tr(\rho) = 1$

$Tr(\rho)$ for mixed state

$$Tr(\rho) = Tr\left(\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|\right)$$

Using eq (1)

$$Tr(\rho) = \sum_{i=1}^{n} p_i Tr(|\psi_i\rangle\langle\psi_i|)$$

Using eq (2)

$$Tr(\rho) = \sum_{i=1}^{n} p_i \langle\psi_i|\psi_i\rangle$$

$$Tr(\rho) = \sum_{i=1}^{n} p_i = 1$$

Properties of Trace

$$Tr(\alpha A) = \alpha Tr(A) \quad \text{......(1)}$$

$$Tr(|\psi\rangle\langle\phi|) = \langle\psi|\phi\rangle \quad \text{......(2)}$$

$$Tr(A+B) = Tr(A) + Tr(B) \quad \text{......(3)}$$

$Tr(\rho) = 1$

For both pure and mixed sates

$Tr(\rho^2)$ for a pure state

$$\rho = |\psi\rangle\langle\psi|$$

$$\boxed{Tr(\rho) = 1}$$

$$\rho^2 = \rho\rho = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)$$

$$\rho^2 = \rho\rho = |\psi\rangle(\langle\psi|\psi\rangle)\,\langle\psi|$$

$$\rho^2 = \rho\rho = |\psi\rangle\langle\psi|$$

$$\rho^2 = \rho\rho = \rho$$

$$\boxed{Tr(\rho^2) = Tr(\rho) = 1}$$ For pure state

$$\boxed{Tr(\rho^2) < 1}$$ For mixed state

Important properties of density of operator

An operator ρ is a density operator if and only if it satisfies the following three requirements:

- The density operator is Hermitian, meaning $\rho = \rho^\dagger$.
- $Tr(\rho) = 1$.
- ρ is a positive operator, meaning $\langle u|\rho|u\rangle \geq 0$ for any state vector $|u\rangle$.

Recall that an operator is positive if and only if it is Hermitian and has nonnegative eigenvalues.

Example 5.2

A system is in the state $|\psi\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + i\sqrt{\frac{2}{3}}|u_2\rangle$, where the $|u_k\rangle$ constitute an orthonormal basis. Write down the density operator, and show it has unit trace.

Solution

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \left[\frac{1}{\sqrt{3}}|u_1\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|u_2\rangle\right]\left[\frac{1}{\sqrt{3}}\langle u_1| - i\frac{\sqrt{2}}{\sqrt{3}}\langle u_2|\right]$$

$$\rho = \frac{1}{3}\left|u_1\rangle\langle u_1\right| - i\frac{\sqrt{2}}{3}\left|u_1\rangle\langle u_2\right| + i\frac{\sqrt{2}}{3}|u_2\rangle\langle u_1| + \frac{2}{3}|u_2\rangle\langle u_2|$$

$$Tr(\rho) = \sum_{j=1}^{n}\langle u_j|\rho|u_j\rangle$$

$$Tr(\rho) = \langle u_1|\rho|u_1\rangle + \langle u_2|\rho|u_2\rangle$$

$$Tr(\rho) = \frac{1}{3} + \frac{2}{3} = 1$$

OR

$$\rho = \begin{pmatrix} \frac{1}{3} & -i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \frac{2}{3} \end{pmatrix} \Rightarrow Tr(\rho) = \frac{1}{3} + \frac{2}{3} = 1$$

Example

$$|\psi\rangle = \frac{1}{2}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|u_3\rangle,$$

$$\rho = \begin{pmatrix} 1/4 & 1/2\sqrt{2} & 1/4 \\ 1/2\sqrt{2} & 1/2 & 1/2\sqrt{2} \\ 1/4 & 1/2\sqrt{2} & 1/4 \end{pmatrix}$$

Density operator of pure and mixed states

Pure State

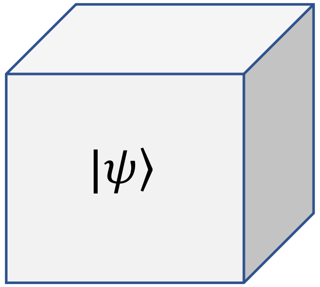
$$\rho = |\psi\rangle\langle\psi|$$

Mixed State

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

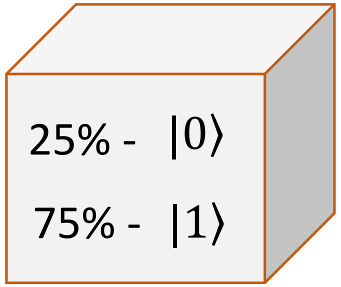
$$\rho = p_1\, |\psi_1\rangle\langle\psi_1| + p_2\, |\psi_2\rangle\langle\psi_2| + p_3\, |\psi_3\rangle\langle\psi_3|$$

Consider two simple systems



Quantum system in state $|\psi\rangle$

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$



Classical ensemble

Both systems are same or different ?

$$\rho = 1/4\; |0\rangle\langle 0| + 3/4\; |1\rangle\langle 1|$$

Coherence

$$\rho = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$$

$$\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

Therefore, systems are different

Time Evolution of Density operator

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$|\psi(t)\rangle = exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi(0)\rangle$$

Time evolution of state $|\psi\rangle$

$$U = exp\left(-\frac{i}{\hbar}\hat{H}t\right) = e^{-\frac{i}{\hbar}\hat{H}t}$$

Time evolution unitary operator

Since $H = H^\dagger$, we can also write

$$-i\hbar\frac{d}{dt}\langle\psi| = \langle\psi|H$$

Now consider derivative (time evolution) of density operator

$$\frac{d\rho}{dt} = \frac{d}{dt}(|\psi\rangle\langle\psi|) = \left(\frac{d}{dt}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\frac{d}{dt}\langle\psi|\right)$$

$$\frac{d\rho}{dt} = \left(\frac{H}{i\hbar}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\langle\psi|\frac{H}{-i\hbar}\right) = \frac{H}{i\hbar}\rho - \rho\frac{H}{i\hbar} = \frac{1}{i\hbar}[H,\rho]$$

$$i\hbar\frac{d\rho}{dt} = [H,\rho]$$

$$\rho(t) = U\rho(t_o)U^\dagger$$

Where U is an unitary operator

Important properties of density of operator

An operator ρ is a density operator if and only if it satisfies the following three requirements:

- The density operator is Hermitian, meaning $\rho = \rho^\dagger$.
- $Tr(\rho) = 1$.
- ρ is a positive operator, meaning $\langle u|\rho|u\rangle \geq 0$ for any state vector $|u\rangle$.

Recall that an operator is positive if and only if it is Hermitian and has nonnegative eigenvalues.

Pure state

$$\langle\psi|\rho|\psi\rangle = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = |\langle\psi|\psi\rangle|^2$$

Mixed state

$$\langle\phi|\rho|\phi\rangle = \sum_{i=1}^{n} p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_{i=1}^{n} p_i |\langle\phi|\psi_i\rangle|^2$$

Example 5.2

Consider the state

$$|a\rangle = \begin{pmatrix} e^{-i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$$

Is $\rho = |a\rangle\langle a|$ a density operator?

Solution

In the $\{|0\rangle, |1\rangle\}$ basis, the state is written as

$$|a\rangle = \begin{pmatrix} e^{-i\phi}\sin\theta \\ \cos\theta \end{pmatrix} = e^{-i\phi}\sin\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-i\phi}\sin\theta|0\rangle + \cos\theta|1\rangle$$

$$\rho = |a\rangle\langle a| = (e^{-i\phi}\sin\theta|0\rangle + \cos\theta|1\rangle)(e^{i\phi}\sin\theta\langle 0| + \cos\theta\langle 1|)$$

$$= \sin^2\theta|0\rangle\langle 0| + e^{-i\phi}\sin\theta\cos\theta|0\rangle\langle 1| + e^{i\phi}\sin\theta\cos\theta|1\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1|$$

$$= \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

$$\rho^\dagger = (\rho^T)^* = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

Therefore ρ is Hermitian

1st condition

Example 5.2 (contd.)

$$\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

$$Tr(\rho) = sin^2\theta + cos^2\theta = 1$$

2nd condition

Now for third condition, consider a general state

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle\psi|\rho|\psi\rangle = (a \quad b)\begin{pmatrix} sin^2\theta & e^{-i\phi}sin\theta cos\theta \\ e^{i\phi}sin\theta cos\theta & cos^2\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle\psi|\rho|\psi\rangle = |a|^2\sin^2\theta + ab^*e^{i\phi}\sin\theta\cos\theta + a^*be^{-i\phi}\sin\theta\cos\theta + |b|^2\cos^2\theta$$

If we substitute

$$z = ae^{-i\phi}\sin\theta$$
$$w = b\cos\theta$$

$\Rightarrow$

$$zz^* = |a|^2\sin^2\theta$$
$$ww^* = |b|^2\cos^2\theta$$

and

$$wz^* = ab^*e^{i\phi}\sin\theta\cos\theta$$
$$zw^* = a^*be^{-i\phi}\sin\theta\cos\theta$$

Above expression for expectation value is similar to

$$(z+w)(z^*+w^*) = zz^* + wz^* + w^*z + ww^* = |z+w|^2 \geq 0$$

Therefore

$$\langle\psi|\rho|\psi\rangle = |ae^{-i\phi}\sin\theta + b\cos\theta|^2 \geq 0$$

$\langle\psi|\rho|\psi\rangle \geq 0$ mean ρ is a positive operator

3rd condition

Probability of obtaining a given measurement result

Review of projection operator

$$|\psi\rangle = \sum_n a_n u_n\rangle$$

$$|\psi\rangle = a_1|u_1\rangle + a_2|u_2\rangle + \cdots a_n|u_n\rangle$$

Consider a **projection operator**

$$P_i = |u_i\rangle\langle u_i|$$

$$P_i|\psi\rangle = a_i|u_i\rangle$$

Prob. of measuring a_i state

$$\mathrm{Pr}(a_i) = |P_i|\psi\rangle|^2 = \langle\psi|P_i^\dagger P_i|\psi\rangle = \langle\psi|P_i^2|\psi\rangle = \langle\psi|P_i|\psi\rangle$$

Now probability of measuring $a_i$state using P_i projection operator

$$Pr(a_i) = |Pi|\psi\rangle|^2 = \langle\psi|P_i^\dagger P_i|\psi\rangle$$

$$Pr(a_i) = \langle\psi|P_iP_i|\psi\rangle = \langle\psi|P_i^2|\psi\rangle$$

Projection operator is Hermitian

$$Pr(a_i) = \langle\psi|P_i|\psi\rangle$$

$$Pr(a_i) = \langle\psi|P_i|\psi\rangle$$

$$Pr(a_i) = \langle\psi|P_i|\psi\rangle = \mathrm{Tr}(P_i|\psi\rangle\langle\psi|)$$

Using a property of trace

New $|\psi'\rangle$ state after measurement using projection operator P_i

$$P_i|\psi\rangle \equiv |\psi'\rangle$$

$$|\psi\rangle \to \frac{P_i|\psi\rangle}{\langle\psi'|\psi'\rangle} = \frac{P_i|\psi\rangle}{\langle\psi|P_i^\dagger P_i|\psi\rangle} = \frac{P_i|\psi\rangle}{\langle\psi|P_i|\psi\rangle}$$

$$P_i^\dagger P_i = P_i\, P_i = P_i^2 = P_i$$

Probability of obtaining a given measurement result using density operator

$$p(a_n) = \langle u_n|\rho|u_n\rangle = Tr(|u_n\rangle\langle u_n|\rho) = Tr(P_n\rho) \tag{5.9}$$

$$P(m) = Tr(M_m^\dagger M_m\rho) \tag{5.10}$$

$$\rho \to \frac{P_n\rho P_n}{Tr(P_n\rho)} \tag{5.11}$$

$$\frac{M_m\rho M_m^\dagger}{Tr(M_m^\dagger M_m\rho)} \tag{5.12}$$

Example 5.3

Does the matrix

$$\rho = \begin{pmatrix} \frac{1}{4} & \frac{1-i}{4} \\ \frac{1-i}{4} & \frac{3}{4} \end{pmatrix}$$

represent a density operator?

Solution

$$Tr(\rho) = \frac{1}{4} + \frac{3}{4} = 1$$

$$\rho^{\dagger} = \begin{pmatrix} \frac{1}{4} & \frac{1+i}{4} \\ \frac{1+i}{4} & \frac{3}{4} \end{pmatrix}$$

$$\rho \neq \rho^{\dagger}$$

Therefore ρ can not represent a density matrix

Example 5.4

A system is found to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

(a) Write down the density operator for this state.

(b) Write down the matrix representation of the density operator in the $\{|0\rangle,|1\rangle\}$ basis. Verify that $Tr(\rho)=1$, and show this is a pure state.

(c) A measurement of Z is made. Calculate the probability that the system is found in the state $|0\rangle$ and the probability that the system is found in the state $|1\rangle$.

(d) Find $\langle X\rangle$.

Solution

$$\rho = |\psi\rangle\langle\psi| = \left[\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right]\left[\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right] =$$

$$\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \quad \Rightarrow \quad Tr(\rho) = \frac{1}{5} + \frac{2}{5} = 1$$

$$\rho^2 = \rho\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{25} + \frac{4}{25} & \frac{2}{25} + \frac{8}{25} \\ \frac{2}{25} + \frac{8}{25} & \frac{4}{25} + \frac{16}{25} \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} \frac{5}{25} & \frac{10}{25} \\ \frac{10}{25} & \frac{20}{25} \end{pmatrix} \Rightarrow Tr(\rho^2) = \frac{5}{25} + \frac{20}{25} = 1$$

pure state

d) Probability of measuring $|0\rangle$ *and* $|1\rangle$states

$$p_{|0\rangle} = Tr(P_0\rho)$$

$$p(a_n) = \langle u_n|\rho|u_n\rangle = Tr(|u_n\rangle\langle u_n|\rho) = Tr(P_n\rho)$$

$$p_{|0\rangle} = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{5} \Rightarrow$$

Similarly $p_{|1\rangle} = Tr(P_1\rho) = \frac{4}{5}$

e) $\langle A\rangle = Tr(\rho A) \Rightarrow \langle X\rangle = Tr(\rho X) = \left[\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$

Example 5.5

Consider a statistical mixture; where 75% is $|+\rangle$ and 25% is $|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

What is the probability of finding the mixture in $|0\rangle$ and $|1\rangle$?

Solution

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho = \frac{75}{100}|+\rangle\langle+| + \frac{25}{100}|-\rangle\langle-|$$

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \rho^2 = \rho\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{9}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$$

$$Tr(\rho^2) = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}$$

ρ represents a mixed state

Example 5.5 contd.

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-| \quad \Rightarrow \quad \rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Now express ρ in $|0\rangle$ and $|1\rangle$

$$\rho = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)((|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|))$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Now

$$p_{|0\rangle} = Tr(P_0\rho) = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$$

Similarly find

$$p_{|1\rangle} = Tr(P_1\rho)$$

Example 5.5 contd.

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

Statistical mixture; where 75% is $|+\rangle$ and 25% is $|-\rangle$

Do they represent the same system?

Now consider a pure state

$$|\psi\rangle = \sqrt{\frac{3}{4}}|+\rangle + \sqrt{\frac{1}{4}}|-\rangle$$

Prob of measuring $|+\rangle$ is 75%

Prob of measuring $|-\rangle$ is 25%

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

In $|+\rangle, |-\rangle$ basis

Now consider

$$|\psi\rangle = \sqrt{\frac{3}{4}}|+\rangle + \sqrt{\frac{1}{4}}|-\rangle$$

In $|0\rangle, |1\rangle$ basis

$$|\psi\rangle = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)|0\rangle + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)|1\rangle$$

$$\Rightarrow p_{|0\rangle} = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2 = \frac{2+\sqrt{3}}{4} = 0.93$$

Example 5.6

Example 5.7

Solve these examples yourself

Example 5.8

Example 5.9

Example 5.10

Example 5.9

Suppose

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{i}{3} \\ \frac{-i}{3} & \frac{2}{3} \end{pmatrix}$$

(a) Show that ρ is Hermitian and has positive eigenvalues that satisfy $0 \leq \lambda_i \leq 1$, and $\sum \lambda_i = 1$.

(b) Is this a mixed state?

(c) Find $\langle X \rangle$ for this state.

Example 5.10

Consider an ensemble in which 40% of the systems are known to be prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

and 60% of the systems are prepared in the state

$$|\phi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

(a) Find the density operators for each of these states, and show they are pure states. If measurements are made on systems in each of these states, what are the probabilities they are found to be in states $|0\rangle$ and state $|1\rangle$, respectively?

(b) Determine the density operator for the ensemble.

$\rho = 0.4|\psi\rangle\langle\psi| + 0.6|\phi\rangle\langle\phi|$

(c) Show that $Tr(\rho) = 1$.

(d) A measurement of Z is made on a member drawn from the ensemble. What are the probabilities it is found to be in state $|0\rangle$ and state $|1\rangle$, respectively?

$$\rho_1 = |\psi\rangle\langle\psi| = \frac{1}{4}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$$

$$\rho_2 = |\phi\rangle\langle\phi| = \frac{1}{4}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$$

$$\rho = 0.4|\psi\rangle\langle\psi| + 0.6|\phi\rangle\langle\phi|$$

$$\rho = \frac{2}{5}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} + \frac{3}{5}\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$$

$$\rho = \begin{pmatrix} \frac{17}{60} & \frac{(8\sqrt{2}+9\sqrt{3})}{60} \\ \frac{(8\sqrt{2}+9\sqrt{3})}{60} & \frac{43}{60} \end{pmatrix}$$

$$Tr(\rho) = \frac{17}{60} + \frac{43}{60} = 1$$

$$P_{|0\rangle} = Tr(P_0\rho)$$

OR

$$P_{|0\rangle} = \langle 0|\rho|0\rangle$$

$$P_{|0\rangle} = Tr\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix}\frac{17}{60} & \frac{(8\sqrt{2}+9\sqrt{3})}{60}\\ \frac{(8\sqrt{2}+9\sqrt{3})}{60} & \frac{43}{60}\end{pmatrix}$$

$$P_{|0\rangle} = \frac{17}{60}$$

Completely Mixed State

$$\rho = \frac{1}{n} I$$

$$\rho^2 = \frac{1}{n^2} I.$$

$$Tr(\rho^2) = \frac{1}{n^2} Tr(I) = \frac{1}{n^2}(n) = \frac{1}{n}$$

Partial trace and reduced density matrix

5.4 Reduced Density Operators

Ch -5 (Book by Bernard)

Consider a direct product state of two qubits,

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \,. \tag{5.26}$$

$$\boldsymbol{\rho}_{AB} = |\psi_{AB}\rangle \langle\psi_{AB}| = |\psi_A\rangle \langle\psi_A| \otimes |\psi_B\rangle \langle\psi_B| \tag{5.27}$$

You can verify that

$$Tr\, \boldsymbol{\rho}^2_{AB} = 1 \tag{5.28}$$

Partial Trace

$$\begin{aligned} Tr_B\, \boldsymbol{\rho}_{AB} &\equiv |\psi_A\rangle \langle\psi_A|\; Tr(|\psi_B\rangle \langle\psi_B|) \\ Tr_A\, \boldsymbol{\rho}_{AB} &\equiv Tr(|\psi_A\rangle \langle\psi_A|)\;\; |\psi_B\rangle \langle\psi_B| \,. \end{aligned} \tag{5.29}$$

For pure sate $Tr(|\psi_B\rangle \langle\psi_B|) = 1$

Therefore

$$Tr_B\, \boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A$$

$$Tr_A\, \boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_B$$

Where $\boldsymbol{\rho}_A = |\psi_A\rangle \langle\psi_A| \quad \boldsymbol{\rho}_B = |\psi_B\rangle \langle\psi_B|$

Reduced single qubit density matrix

Entangled State

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \Big) = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle \Big) \qquad (5.33)$$

Density Operator for above entangled state

$$\boldsymbol{\rho}_{AB} = |\psi_{AB}\rangle \langle\psi_{AB}| = \frac{1}{2}\Big(|01\rangle \langle 10| + |01\rangle \langle 01| + |10\rangle \langle 10| + |10\rangle \langle 01| \Big)(5.34)$$

$$\boldsymbol{\rho}^2_{AB} = |\psi_{AB}\rangle \langle\psi_{AB}|\, |\psi_{AB}\rangle \langle\psi_{AB}| = |\psi_{AB}\rangle \langle\psi_{AB}|$$

$$Tr\, \boldsymbol{\rho}^2_{AB} = Tr\, |\psi_{AB}\rangle \langle\psi_{AB}| = 1$$

Therefore ρ_{AB} represents a pure state

$$Tr_B\, \boldsymbol{\rho}_{AB} =$$

$$\boldsymbol{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{2}\Big(|01\rangle\langle 10| + |01\rangle\langle 01| + |10\rangle\langle 10| + |10\rangle\langle 01| \Big) \tag{5.34}$$

$$\boldsymbol{\rho}_{A} \equiv Tr_B\, \boldsymbol{\rho}_{AB} = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big).$$

$$\boldsymbol{\rho}_A \equiv Tr_B\, \boldsymbol{\rho}_{AB} = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big).$$

$$\boldsymbol{\rho}_A^2 = \frac{1}{4}\Big(|0\rangle\langle 0|\; |0\rangle\langle 0| + |0\rangle\langle 0|\; |1\rangle\langle 1| + |1\rangle\langle 1|\; |0\rangle\langle 0| + |1\rangle\langle 1|\; |1\rangle\langle 1| \Big)$$

$$\rho_A^2 = \frac{1}{4}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big)$$

$$\rho_A^2 = \frac{1}{2}$$

$$\rho_A^2 < 1$$

Therefore $\rho_A$represents a mixed sate

Density operator and Block Vector

$$\rho = \frac{1}{2}(I + \vec{S} \cdot \vec{\sigma})$$

$\vec{S}$ is called the *Bloch vector*. The magnitude of the Bloch vector satisfies $|\vec{S}| \leq 1$, with equality for pure states. Otherwise, if $|\vec{S}| < 1$, then the state is a mixed state. The components of the Bloch vector are calculated by considering the expectation values of the operators X, Y, and Z. That is,

$$\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z} = \langle X\rangle\hat{x} + \langle Y\rangle\hat{y} + \langle Z\rangle\hat{z}$$

$$S_x = Tr(\rho X), \quad S_y = Tr(\rho Y), \quad S_z = Tr(\rho Z)$$