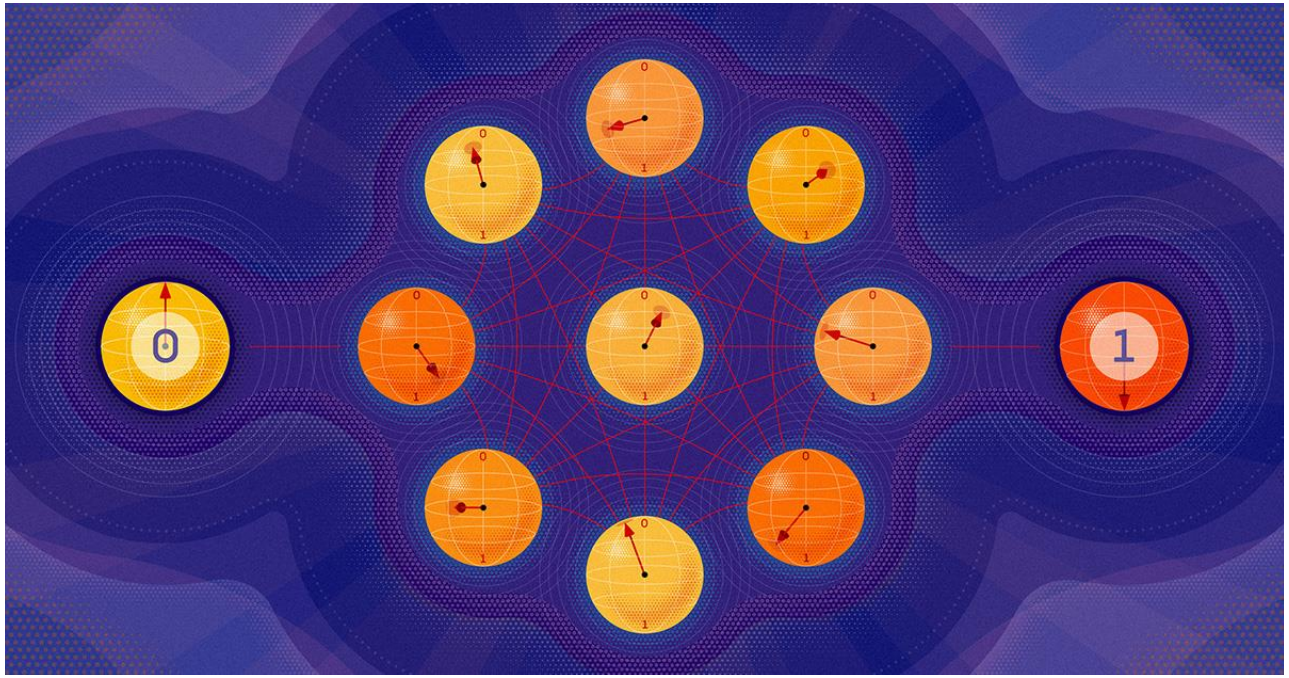
An Introduction to Quantum Error Correction



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From exam point of view please focus on Slides 13-33 and 37-40

References

Quantum Error Correction: An Introductory Guide,
Joschka Roffe, Contemporary Physics (2019)

Quantum Computation and Quantum Information,
M. A Nielsen and I. A. Chuange 10th Anv. Edition (2010)

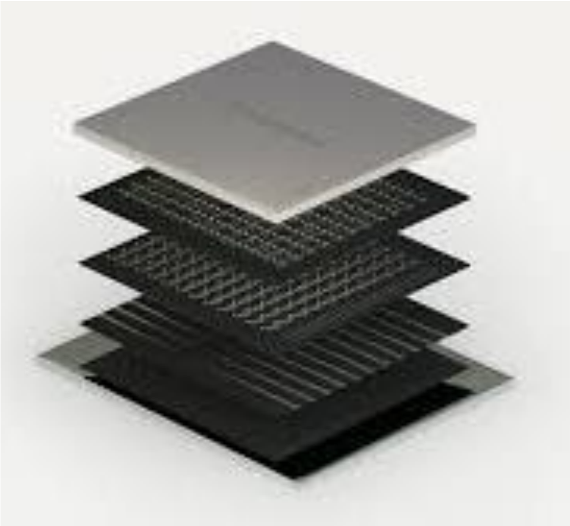
Quantum Error Correction for Beginners, Simon J. Devitt, William J. Munro, and Kae Nemoto, Rep. Prog. Phys. 76 , 076001 2013)

Stabilizer Codes and Quantum Error Correction
PhD Dissertation by Daniel Gottesman (1997)

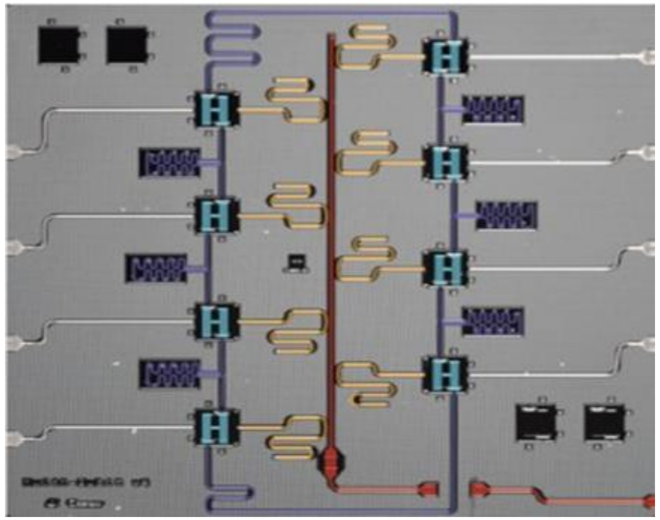
The need of Error Correction

Bit flip error $0 \leftrightarrow 1$ is a reality in classical computer, although probability of flip is quite low but not zero, therefore correction is required, parity error

> 100 qubit IBM processor



Qubit don't live in isolation; they interact with environment and we need to **control them as well** – **all can affect the state of a qubit**



The environment is **constantly trying to look at the state**, a process called *decoherence*. One goal of quantum error correction will be to prevent the environment from looking at the data.

Quantum coherence is quite fragile

Source-IBM

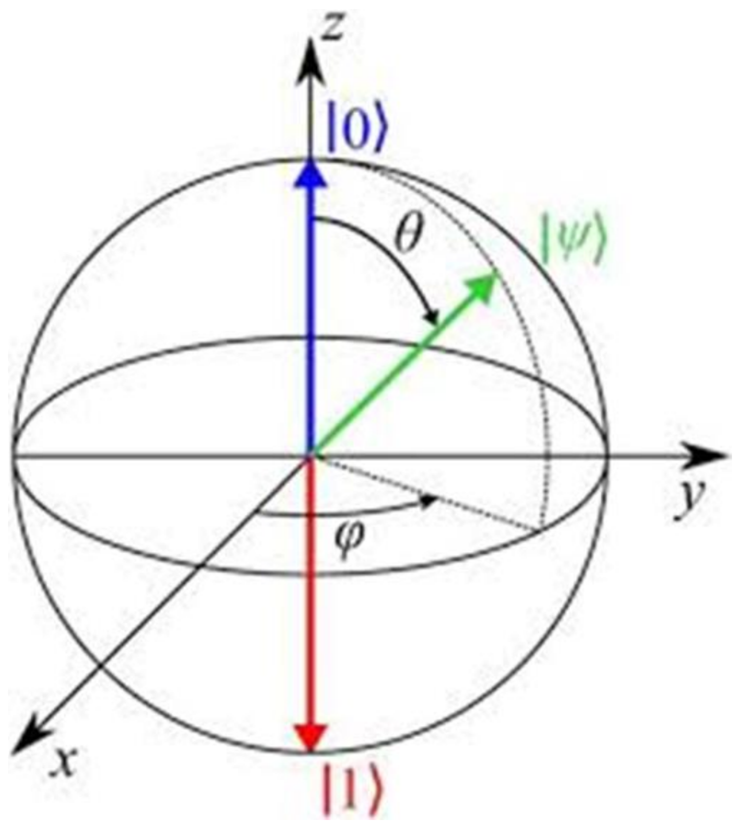
Qubit is NOT Digital like Classical Bit

The best option to visualize the qubit; '**a unit vector on Bloch Sphere**' (state vector $|\psi\rangle$) where θ and φ can have any arbitrary value

Mathematically

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi/2}\sin(\theta/2)|1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



Quantum computers have a continuum of states, so it would seem, at first glance, '**quantum error can not be corrected**'.

Classical computers are digital: after each step, they correct themselves to the closer of 0 or 1 (parity check).

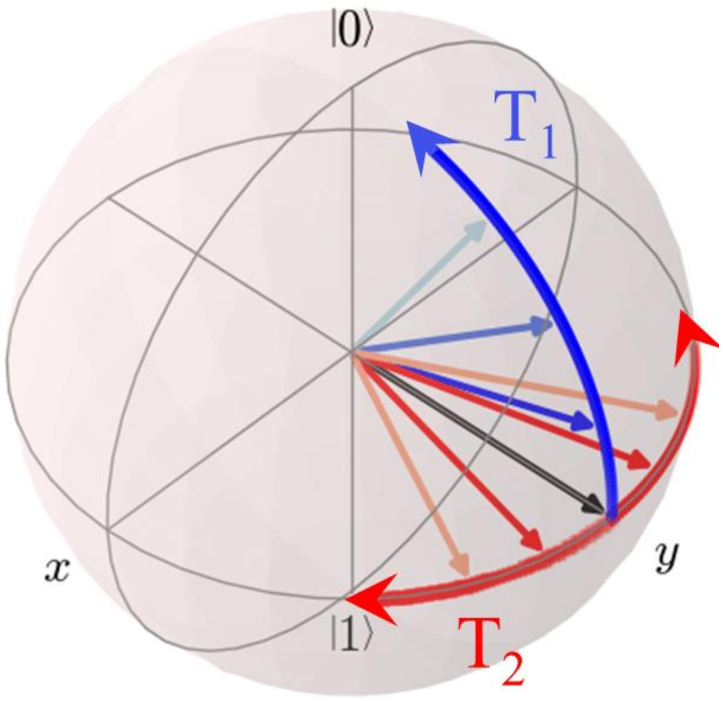
Quantum computers have a great deal of potential, but to realize that potential, they need some sort of protection from noise.

Quantum Error Correction (QEC)?

Physical Qubits fundamentally are NOT error free,

T_1 (Energy Relaxation) **and** T_2 (Phase Noise)

In addition, Control Error



Quantum Gate Error rate $\sim 10^{-2} - 10^{-4}$

Error rate in classical computers are fundamentally very low order of 10^{-20}

In Qubit, required error $\sim 10^{-15} - 10^{-20}$

QEC is more than parity check in classical computer!

Why do we need QEC

Algorithms like Shor, Grover, etc. assume that qubits are perfect
We run them with imperfect physical qubit, therefore out put can/will be nonsense – to get required output we need qubits that are flawless

Qubits don't work exactly the way they should

- Each gate is slightly wrong
- Qubits get poked by (small) external forces

Any qubit made of a real-life physical system will be at least bit rubbish

Somehow how we need '**perfect qubit**' **out of noisy physical qubits**

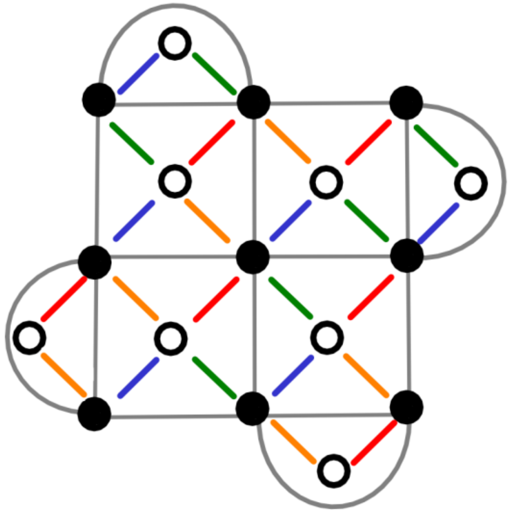
Such perfect qubits are called Logical qubits

Noisy qubit → perfect qubit

In QEC is how we achieve this

Typically **many physics qubits are needed per logical qubit**

A logical qubit, made of array of 17 physical qubit
(NOT a Perfect logical qubit, however better than single physical qubit)



The goal of QEC is to **create logical qubits** that are **less noisy compared to available physical qubits**. For this, the information of a quantum bit needs to be encoded such that it is hidden from the environment and can be decoded later without errors.

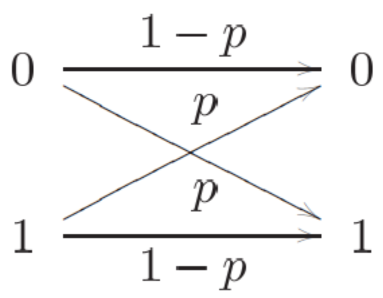
Classical Error Correction to Quantum Error Correction (QEC)

Three bit error correction
(repetition code)

$0 \rightarrow 000$

$1 \rightarrow 111$

Bit strings 000 and 111 are called referred to as the ***logical* 0** and ***logical* 1**



Probability that two or more of the bits are flipped is

$$3p^2(1-p)+p^3 \;\Rightarrow p_e = 3p^2 - 2p^2$$

$$p_e < p, whenever\ p < 1/2$$

In case of single bit flip, based upon 'majority voting' error can be detected/corrected

$0 \rightarrow 010 \quad \rightarrow \mathbf{000}$

$1 \rightarrow 101 \quad \rightarrow \mathbf{111}$

Majority voting fails if two or more of the bits were flipped

If the probability of one bit flip is less than 1/2, then probability of flipping of 2 or 3 qubit will be very small

QEC is Different from Classical Error

Although the field of **QEC is largely based on classical coding theory**, there are **several issues that need to be considered when transferring classical error correction techniques to the quantum regime.**

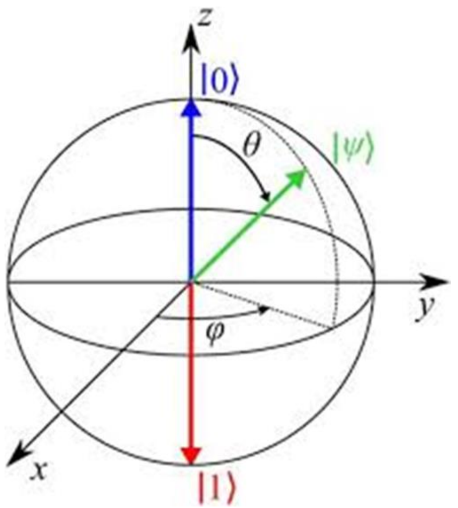
Direct measurement cannot be used to effectively protect against errors, **since this will act to destroy any quantum superposition** that is being used for computation. Error correction protocols must therefore be employed, **which can detect and correct errors without determining any information regarding the qubit state.**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Unlike classical information, qubits are susceptible to both traditional **bit errors** $|0\rangle \leftrightarrow |1\rangle$, and also **phase errors** $|0\rangle - |1\rangle \leftrightarrow |0\rangle + |1\rangle$. Hence any **error correction procedure needs to be able to simultaneously correct for both**.

QEC is Different from Classical Error

Errors in quantum information are intrinsically continuous
(i.e. qubits do not experience full bit or phase flips but rather an angular shift of the qubit state by any angle).



At its most basic level, QEC utilizes the idea of redundant encoding. This is where the **total size of the Hilbert space is expanded beyond what is needed to store a single qubit of information**. This way, errors on individual qubits are mapped to large set of mutually orthogonal subspaces, the size of which is determined by the number of qubits utilized in the code.

The **error correction protocol cannot allow us to gain information regarding the coefficients,** and of the encoded state; as **doing so would collapse the system**.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Classical Error Correction to Quantum Error Correction (QEC)

To protect **quantum state against the effect of noise** we would to develop ***quantum error correcting code***

There are some **important differences between classical information and quantum information** that **require new ideas** to be introduced to make such quantum error correcting code possible

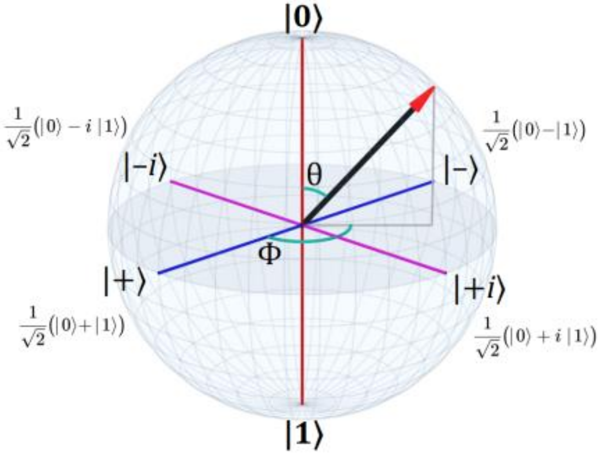
Classically **bit repetition → errors are digital** (bit flip) **→ measure the error → correct the error**

Quantum State can be copied (No Cloning Theorem)

Quantum errors are continuous

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi/2}\sin(\theta/2)|1\rangle$$

A continuum of different errors may occur on a si… determination may require infinite precisio… …ces



Measurement destroys quan… …mation $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

In classical error-correctio… …**serve the output** and then decide the decoding procedure

Upon measurement 'qua… …state' will collapse – **can't read/measure it to correct it**

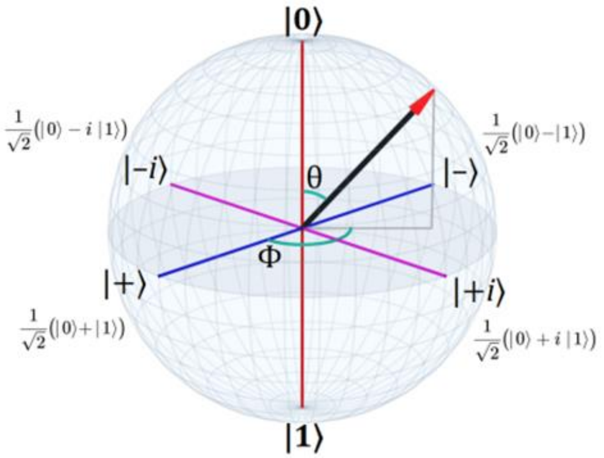
Fortunately none of these problems is fatal

Classical Error Correction to Quantum Error Correction (QEC)

Digitization of Quantum errors

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi/2}\sin(\theta/2)|1\rangle$$

Where $|\cos(\theta/2)|^2 + \left|e^{i\varphi/2}\sin(\theta/2)\right|^2 = 1$



Qubit error can occur by variety of physical processes. The simplest errors cause the qubit to coherently rotate from one point the Bloch to another

$$U(\delta\theta, \delta\varphi)|\psi\rangle = \cos\left(\frac{\theta + \delta\theta}{2}\right)|0\rangle + e^{i(\varphi+\delta\varphi)/2}\sin\left(\frac{\theta + \delta\theta}{2}\right)|1\rangle$$

Single qubit coherent error process described by $U(\delta\theta, \delta\varphi)$ can be expanded in the in terms of Pauli matrices

$$U(\delta\theta, \delta\varphi) = \alpha_I\, \mathbb{I}|\psi\rangle + \alpha_X\hat{X}|\psi\rangle + \alpha_Y\hat{Y}|\psi\rangle + \alpha_Z\hat{Z}|\psi\rangle \;\Rightarrow\; U(\delta\theta, \delta\varphi) = \alpha_I\, \mathbb{I}|\psi\rangle + \alpha_X\hat{X}|\psi\rangle + \alpha_{XZ}\hat{X}\hat{Z}|\psi\rangle + \alpha_Z\hat{Z}|\psi\rangle$$

Quantum error correction process itself involved performing projective measurements that causes the above superposition to collapse to a subset of its terms – **quantum error can be corrected with X- and Z-Pauli matrices** – **referred to as digitization error**

Quantum Error; Bit-flip & Phase-flip

All types of error in a 'quantum state' can be fixed by fixing two types of errors, bit-flip an phase-flip

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Bit flip error

$$\hat{X}|0\rangle = |1\rangle$$
$$\hat{X}|1\rangle = |0\rangle$$

$$\hat{X}|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |\psi\rangle$$

$$\hat{X}|\psi\rangle = \alpha|1\rangle + \beta|0\rangle \neq |\psi\rangle$$

phase flip error

$$\hat{Z}|0\rangle = |0\rangle$$
$$\hat{Z}|1\rangle = -|1\rangle$$

$$\hat{Z}|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \neq |\psi\rangle$$

$$\hat{Z}|\psi\rangle = \alpha|0\rangle - \beta|1\rangle \neq |\psi\rangle$$

There is no classical equivalent of **phase flip**

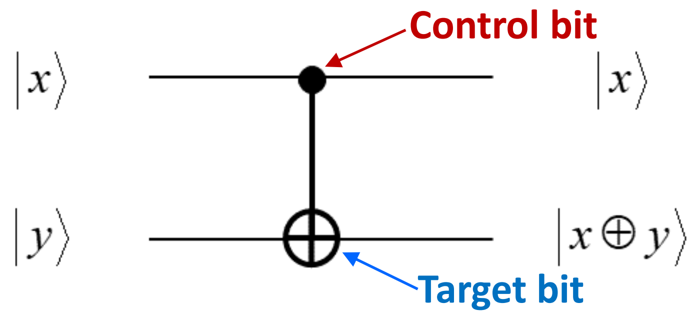
How to fix the bit-flip and phase-flip error?

Quantum Error Correction Codes

	Logical Qubit	Error detection
• **Repetition code**	Repetition of bits	majority voting
• Surface Code	Lattice structure of qubit	$\hat{X}, \hat{Z}$ syndrome measurements
• Bosonic code	Bosonic state of Harmonic Oscillator	parity measurement

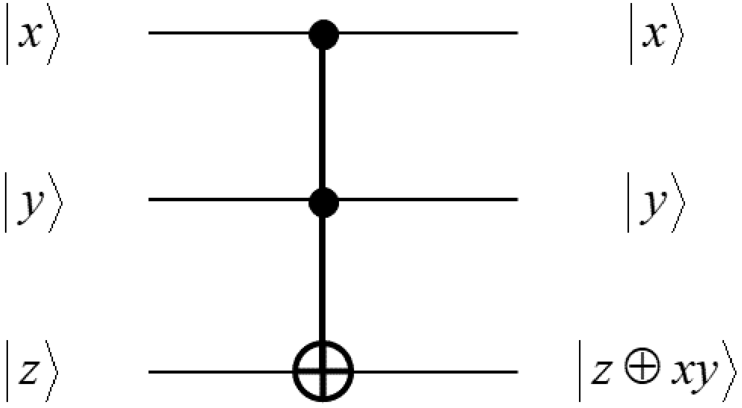
Quantum Gates

CNOT-gate



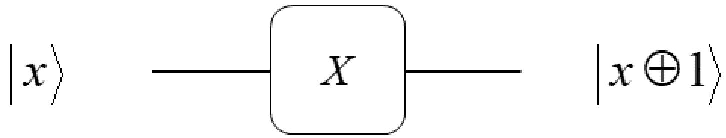
Target bit will flip when **control bit is '1'**

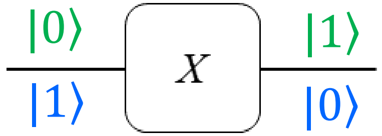
Toffoli-gate



X-gate

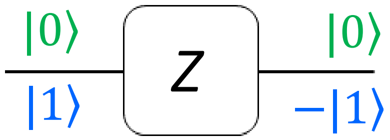
NOT gate



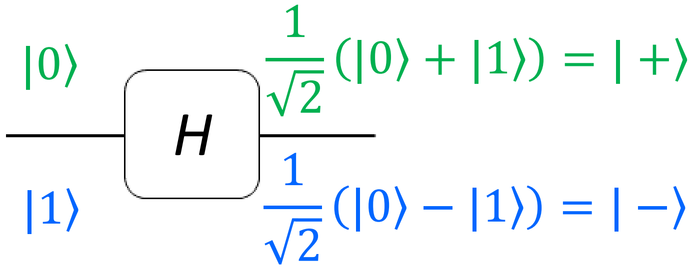


Z-gate

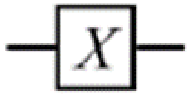
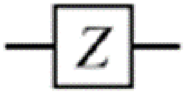
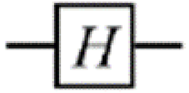
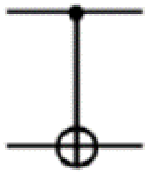
Phase-flip gate



H-gate



Quantum Gates: Symbols and Matric

Gate	Notation	Matrix
NOT (Pauli-X)		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Z		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
CNOT (Controlled NOT)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Quantum Error Correction (bit-flip error)

(repetition code)

Bit flip error

$\hat{X}|0\rangle = |1\rangle$

$\hat{X}|1\rangle = |0\rangle$

To fix bit flip error in quantum state, first we create a **'logical state'**

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ → **Encoded three qubit state** → $|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle = \boldsymbol{\alpha|0\rangle_L + \beta|1\rangle_L}$

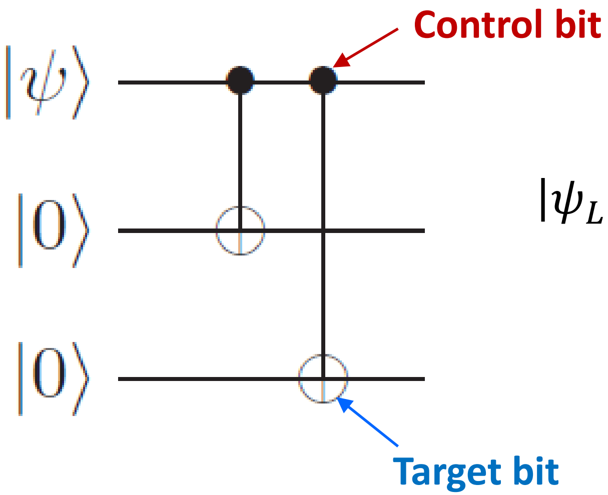
$|0\rangle_L$ - logical $|0\rangle$

$|1\rangle_L$ - logical $|1\rangle$

Following quantum circuit can convert the sate $|\psi\rangle$ to $|\psi\rangle_L$

$|\psi\rangle = \alpha|000\rangle + \beta|100\rangle$

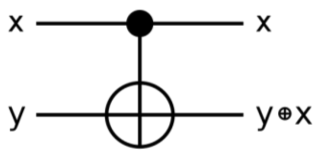
(Input state)



$|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$

(output state)

CNOT gate



input x	input y	output x	output y+x
$|0\rangle$	$|0\rangle$	$|0\rangle$	$|0\rangle$
$|0\rangle$	$|1\rangle$	$|0\rangle$	$|1\rangle$
$|1\rangle$	$|0\rangle$	$|1\rangle$	$|1\rangle$
$|1\rangle$	$|1\rangle$	$|1\rangle$	$|0\rangle$

main idea behind the construction of this three-qubit code was that two orthogonal states remain orthogonal after an error. This condition enables QEC and is known as the *Knill- Laflamme criterion*

Two Stage Error Correction Procedure (bit flip)

(Syndrome Diagnosis) (repetition code)

We have perfectly encoded the state $|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$

Error-detection or Syndrome diagnosis:
measurements which tells us what error, if any, **occurred on the quantum state**. The measurement results the ***error syndrome***

For projection measurements

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$P_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\langle\psi|P_0|\psi\rangle = 1$$

Four bit flip error syndromes (i.e. projection operators)

$P_0 = \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	→	$\langle\psi\|P_0\|\psi\rangle = 1$	No error
$P_1 = \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	→	$\langle\psi\|P_1\|\psi\rangle = 1$	Bit flip on **qubit one**
$P_2 = \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	→	$\langle\psi\|P_2\|\psi\rangle = 1$	Bit flip on qubit two
$P_3 = \|001\rangle\langle 001\| + \|110\rangle\langle 110\|$	→	$\langle\psi\|P_3\|\psi\rangle = 1$	Bit flip on qubit three

Error syndrome contains only information about the **what error has occurred**, **without any information about the state $|\psi\rangle$**

If you know there is a **bit flip error** we can fix it by applying bit flip gate to specific bit

Two Stage Error Correction Procedure (Bit Flip)

(Syndrome Diagnosis – a different approach) (repetition code)

$\hat{Z}|0\rangle = |0\rangle$

$\hat{Z}|1\rangle = -|1\rangle$

Instead of **measuring four projectors**, we perform two measurements, first $\hat{Z}_1\hat{Z}_2$ and $\hat{Z}_2\hat{Z}_3$

$$\hat{Z}_1\hat{Z}_2 = Z \otimes Z \otimes I = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$$

$\hat{Z}_1\hat{Z}_2\,|\psi\rangle_L = \hat{Z}_1\hat{Z}_2[\alpha|000\rangle + \beta|111\rangle]$ Eigen value

+1 no bit flip (or both - Not possible, we assumed)

-1 one of the bit is flipped (**Which one** – we don't know)

Pauli $\hat{Z}$ operates on qubit 1 & 2

$\hat{Z}_2\hat{Z}_3\,|\psi\rangle_L = \hat{Z}_1\hat{Z}_2[\alpha|000\rangle + \beta|111\rangle]$ Eigen value

+1 no bit flip (or both - Not possible, we assumed)

-1 one of the bit is flipped (**Which one** – we don't know)

Now if we combine $\hat{Z}_1\hat{Z}_2$ and $\hat{Z}_2\hat{Z}_3$ measurements results, we can find bit-flip error

Eigen value of $\hat{Z}_1\hat{Z}_2$ and $\hat{Z}_2\hat{Z}_3$ is **negative** – **2nd bit is flipped**

Eigen value of $\hat{Z}_1\hat{Z}_2$ is **negative** and $\hat{Z}_2\hat{Z}_3$ is **positive** – **1st bit is flipped**

$$|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$

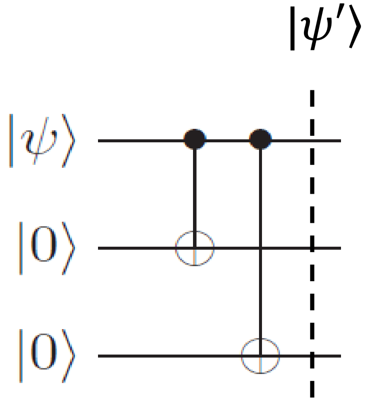
Assuming only one bit can flip, prob of two and three bits flipping is very low

Flipped bit can be fixed by applying **bit flip gate** to a specific bit

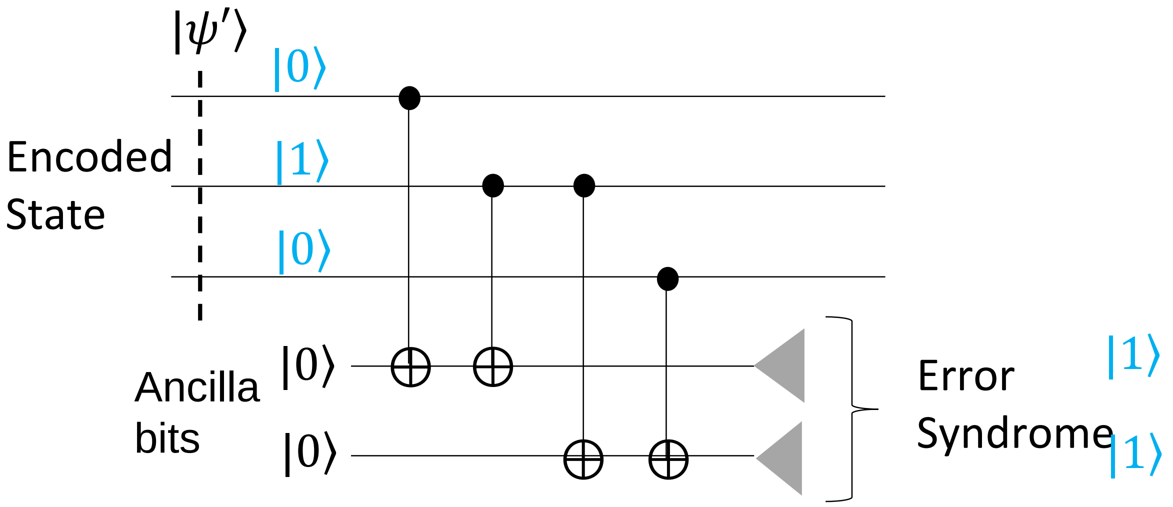
Without knowing the state (i.e. α and β) we **can detect and fix the bit-flip error !**

Bit Flip Error Correction (Syndrome Diagnosis – Circuit based)

(Error deduction using ancilla bits) (repetition code)



$$|\psi'\rangle = |\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$



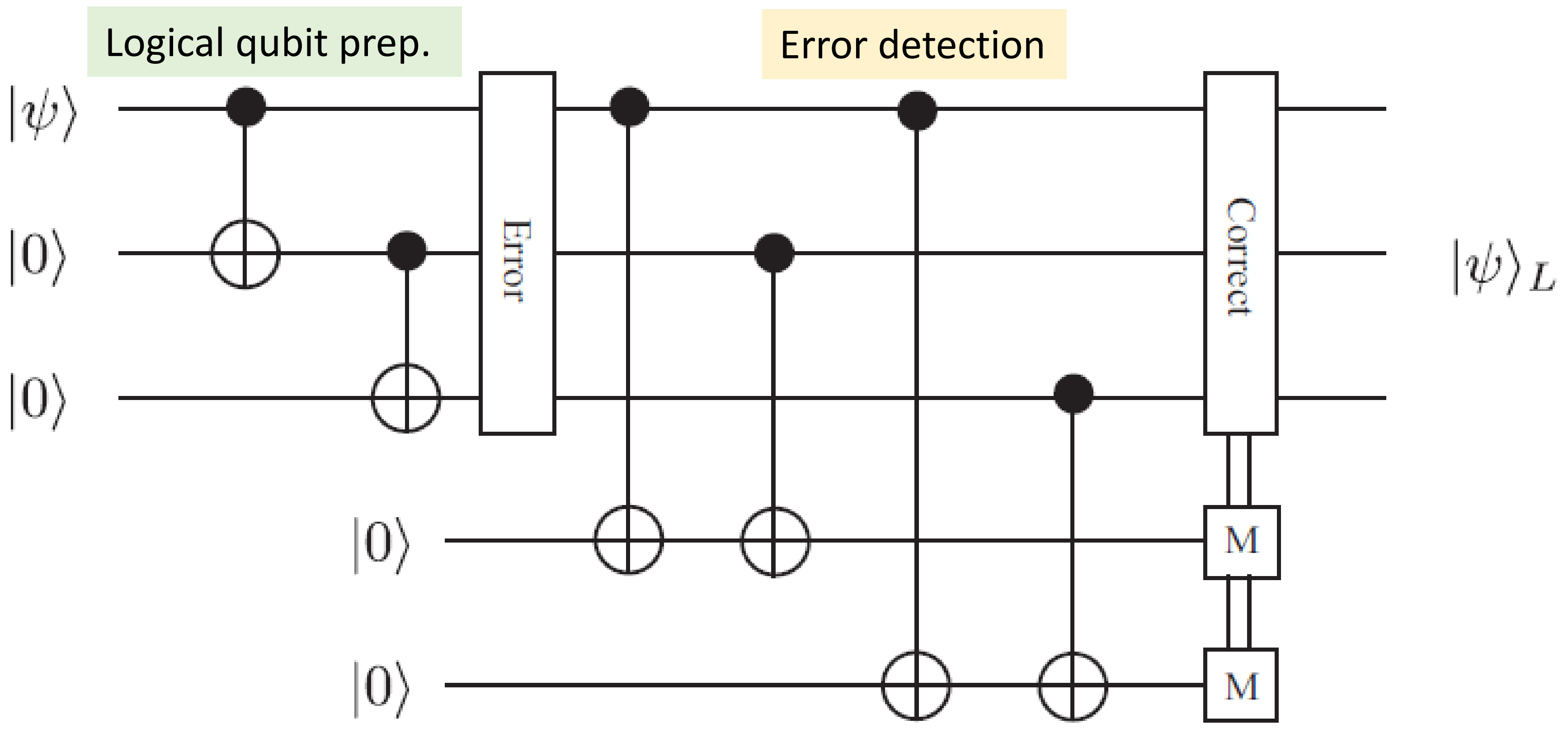
$$|\psi'\rangle = \alpha|000\rangle + \beta|111\rangle$$

Bit Flip	Error Syndrome
No bit flip	00
1st flipped $(\alpha|100\rangle + \beta|011\rangle)$	10
2nd flipped $(\alpha|010\rangle + \beta|101\rangle)$	11
3rd flipped $(\alpha|001\rangle + \beta|110\rangle)$	01

Assuming prob. of more than one bit flip is very low

Flipped bit can fix be fixed by applying ‘bit flip gate’

Bit Flip Error Correction (Syndrome Diagnosis – Circuit based)



Possible final state of five qubits
(before correction)

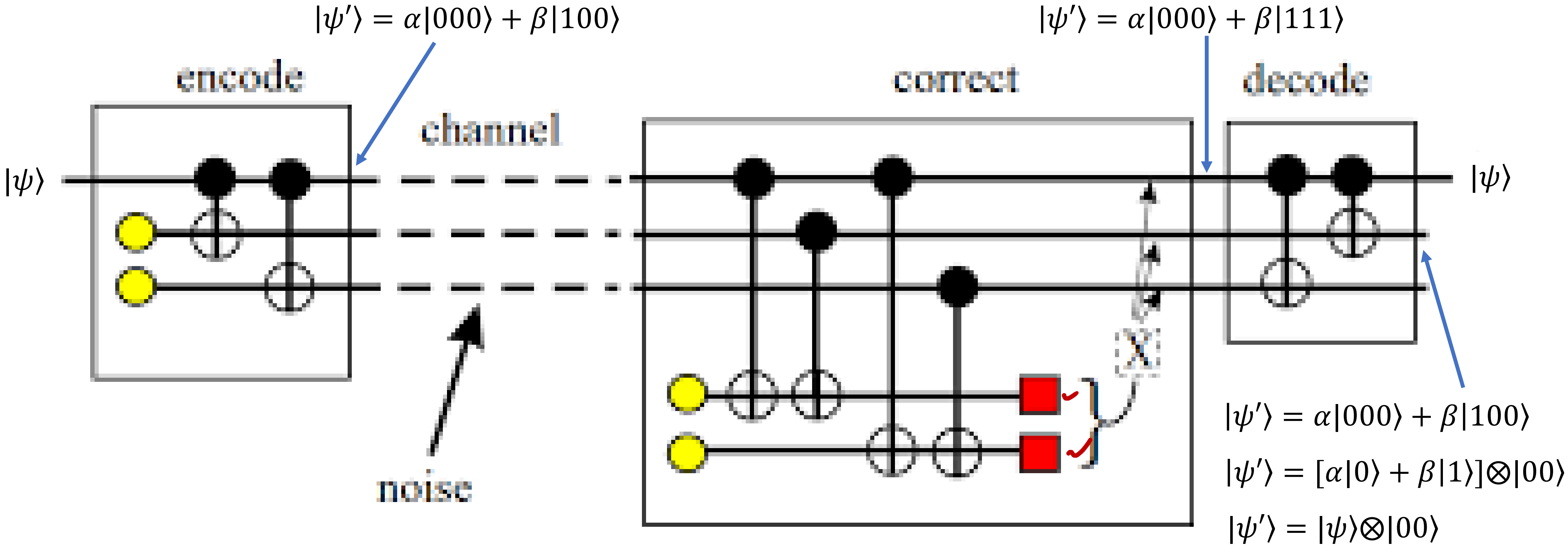
Error Location	Final State, $\|data\rangle \|ancilla\rangle$
No Error	$\alpha \|000\rangle \|00\rangle + \beta \|111\rangle \|00\rangle$
Qubit 1	$\alpha \|100\rangle \|11\rangle + \beta \|011\rangle \|11\rangle$
Qubit 2	$\alpha \|010\rangle \|10\rangle + \beta \|101\rangle \|10\rangle$
Qubit 3	$\alpha \|001\rangle \|01\rangle + \beta \|110\rangle \|01\rangle$

Correction based upon ancilla states

Ancilla Measurement	Collapsed State	Consequence
00	$\alpha \|000\rangle + \beta \|111\rangle$	No Error
01	$\alpha \|001\rangle + \beta \|110\rangle$	$\hat{X}$ on Qubit 3
10	$\alpha \|010\rangle + \beta \|101\rangle$	$\hat{X}$ on Qubit 2
11	$\alpha \|100\rangle + \beta \|011\rangle$	$\hat{X}$ on Qubit 1

Complete Quantum Circuit for bit-flip

(repetition code)



Encoding → error detection → error correction → decoding

All without disturbing the Quantum state $|\psi\rangle$

Phase Flip Error Correction (Syndrome Diagnosis – Circuit based) (repetition code)

Bit Flip $\hat{X}: |0\rangle \leftrightarrow |1\rangle$

Phase Flip $\hat{Z}: |1\rangle = -|1\rangle$ and $|0\rangle = |0\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{\text{Bit flip}} \hat{X}(|\psi\rangle) = \hat{X}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Bit flip has no effect on the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{\text{Phase flip}} \hat{Z}(|\psi\rangle) = \hat{Z}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

This state is **not** same as initial state

Phase flip is purely a **quantum mechanical concept**, no classical equivalent

How to fix the phase flip error?

Phase flip error can turn into 'bit flip'

i.e. in $|+\rangle, |-\rangle$, $\hat{Z}$ **operator act like a bit flip gate**

$\hat{Z}\,|+\rangle = |-\rangle$

$\hat{Z}\,|-\rangle = |+\rangle$

Like a bit flip $|+\rangle \leftrightarrow |-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

For phase-flip we can work in $|+\rangle, |-\rangle$ basis and can use "bit flip" process to fix the 'phase flip error'

Phase Flip Error Correction (Concept) (repetition code)

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

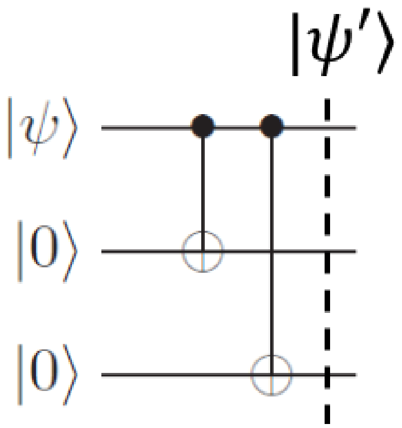
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{\text{Phase flip}} \hat{Z}(|\psi\rangle) = \hat{Z}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

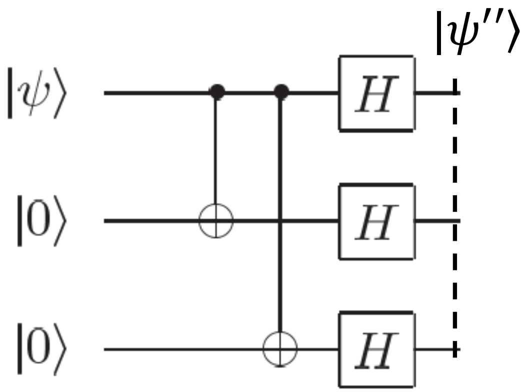
$$|\psi\rangle = |+\rangle \qquad \hat{Z}\,|\psi\rangle = \hat{Z}|+\rangle = |-\rangle$$

Error correction: **coding, error detection, and recovery** – **are performed just as for the bit flip but with respect to $|+\rangle, |-\rangle$ basis instead of $|0\rangle, |1\rangle$ basis**.

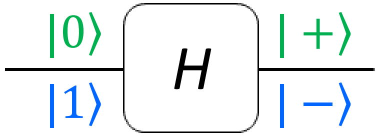
To accomplish this basis change we simply apply the Hadamard gate
(and inverse at appropriate point in the procedure to go back to $|0\rangle, |1\rangle$)



$$|\psi'\rangle = \alpha|000\rangle + \beta|111\rangle$$



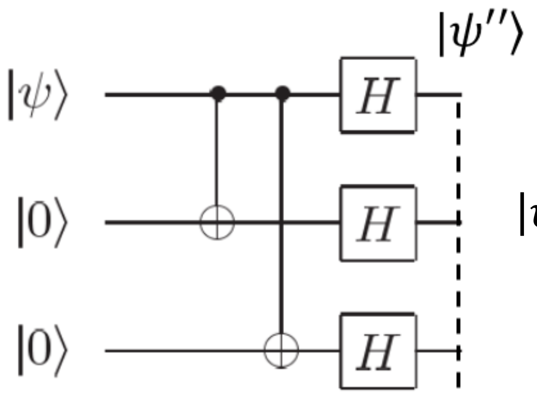
$$|\psi''\rangle = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$$



'Logical state' preparation for bit-flip error

apply Hadamard gate → **Phase flip logical state**

Phase Flip Error Correction (Syndrome Diagnosis – Circuit based) (repetition code)



$$|\psi''\rangle = \alpha|+++\rangle + \beta|---\rangle$$

Equivalently, syndrome measurement may be performed by measuring the observables

$$H^{\otimes 3} P_j H^{\otimes 3} \equiv H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$$

$$H^{\otimes 3} P_j H^{\otimes 3} \equiv H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$

Error detection is achieved by applying the same projective measurements (similar to bit-flip)

$$P_j \rightarrow P'_j \equiv H^{\otimes 3} P_j H^{\otimes 3}$$

Phase flip error is just like bit flip error but in $|+\rangle$ $|-\rangle$ basis

$$\hat{X}|+\rangle = |+\rangle$$

$$\hat{X}|-\rangle = -|-\rangle$$

Just measure the change in sign of 1st and 2nd and 2nd and 3rd qubit to get the phase flipped bit

Phase Flip	**Error Syndrome** X_1X_2 and X_2X_3
No phase flip $(\alpha|+++\rangle + \beta|---\rangle)$	00
1st flipped $(\alpha|-++\rangle + \beta|+--\rangle)$	10
2nd flipped $(\alpha|+-+\rangle + \beta|-+-\rangle)$	11
3rd flipped $(\alpha|++-\rangle + \beta|--+\rangle)$	01

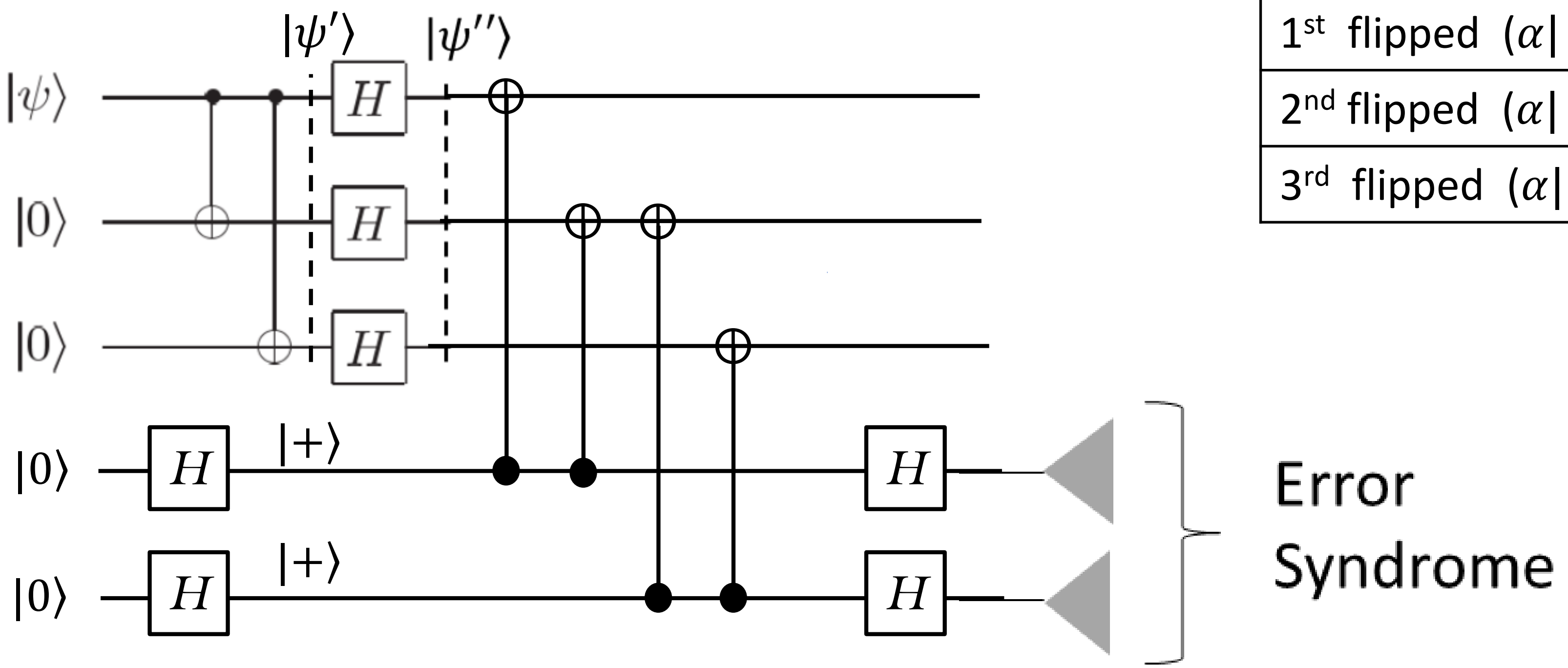
$$X_1X_2|++\rangle = |++\rangle$$

$$X_1X_2|-+\rangle = -|-+\rangle$$

Phase Flip Error Correction (Syndrome Diagnosis – Circuit based)

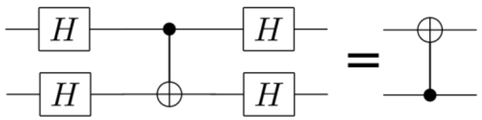
$|\psi'\rangle = \alpha|000\rangle + \beta|111\rangle$

$|\psi''\rangle = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$



Bit Flip	Error Syndrome
No bit flip	00
1st flipped $(\alpha|{-}{+}{+}\rangle + \beta|{+}{-}{-}\rangle)$	10
2nd flipped $(\alpha|{+}{-}{+}\rangle + \beta|{-}{+}{-}\rangle)$	11
3rd flipped $(\alpha|{+}{+}{-}\rangle + \beta|{-}{-}{+}\rangle)$	01

Assuming prob. of more than one bit flip is very low



Flipped phase can fix be fixed by applying '**phase flip gate**'

Syndrome Measurement

$$|++\rangle = \frac{1}{2}[(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)]$$

$$|++\rangle = \frac{1}{2}[(|00\rangle + |01\rangle + |10\rangle + |11\rangle)]$$

$$\text{CNOT}|++\rangle = \tfrac{1}{2}[(|00\rangle + |01\rangle + |11\rangle + |10\rangle)]$$

$$\text{CNOT}|++\rangle = |++\rangle$$

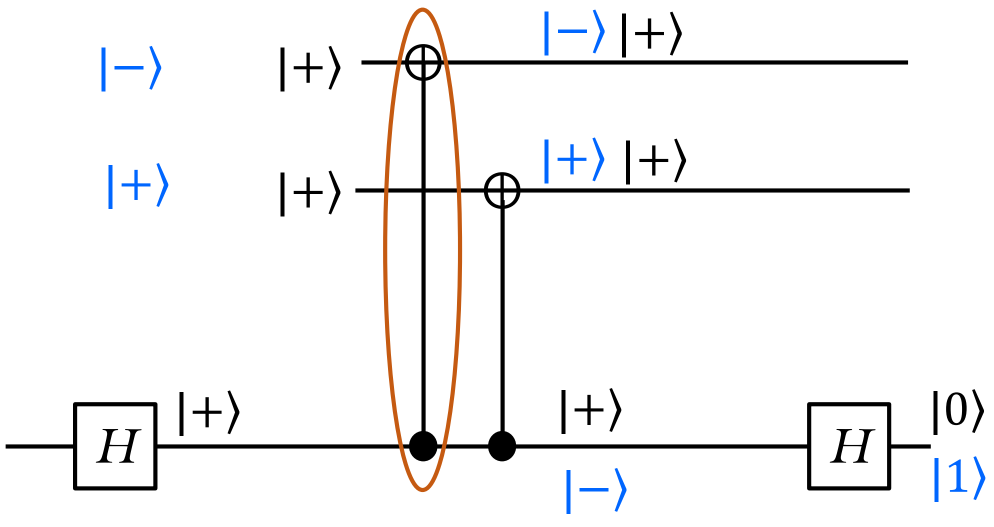
$$|+-\rangle = \frac{1}{2}[(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)]$$

$$|+-\rangle = \frac{1}{2}[(|00\rangle - |01\rangle + |10\rangle - |11\rangle)]$$

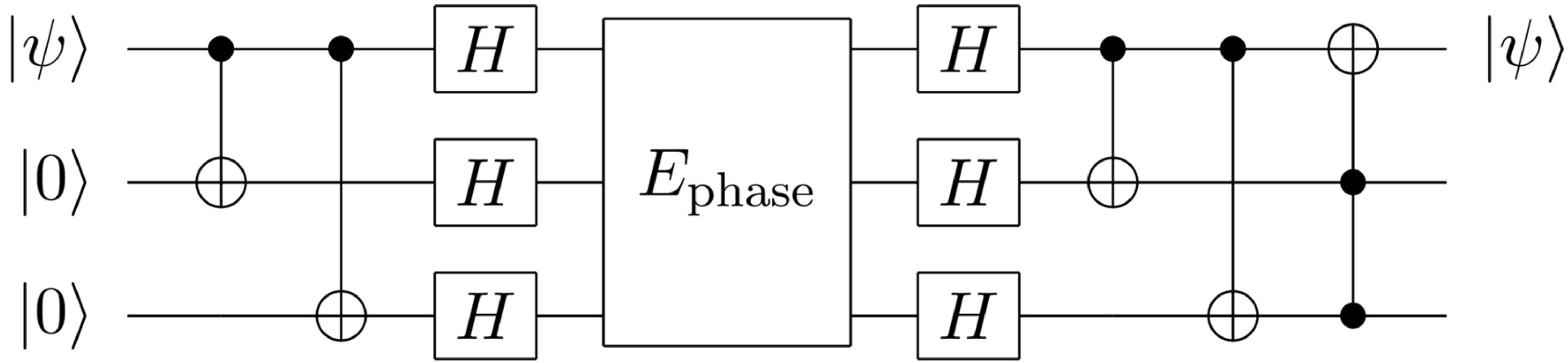
$$\text{CNOT}|+-\rangle = \tfrac{1}{2}[(|00\rangle - |01\rangle + |11\rangle - |10\rangle)]$$

$$\text{CNOT}|+-\rangle = \tfrac{1}{2}[|0\rangle(|0\rangle - |1\rangle) - |1\rangle(-|1\rangle + |0\rangle)]$$

$$\text{CNOT}|+-\rangle = \tfrac{1}{2}[(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)] \Rightarrow \text{CNOT}|+-\rangle = |--\rangle$$

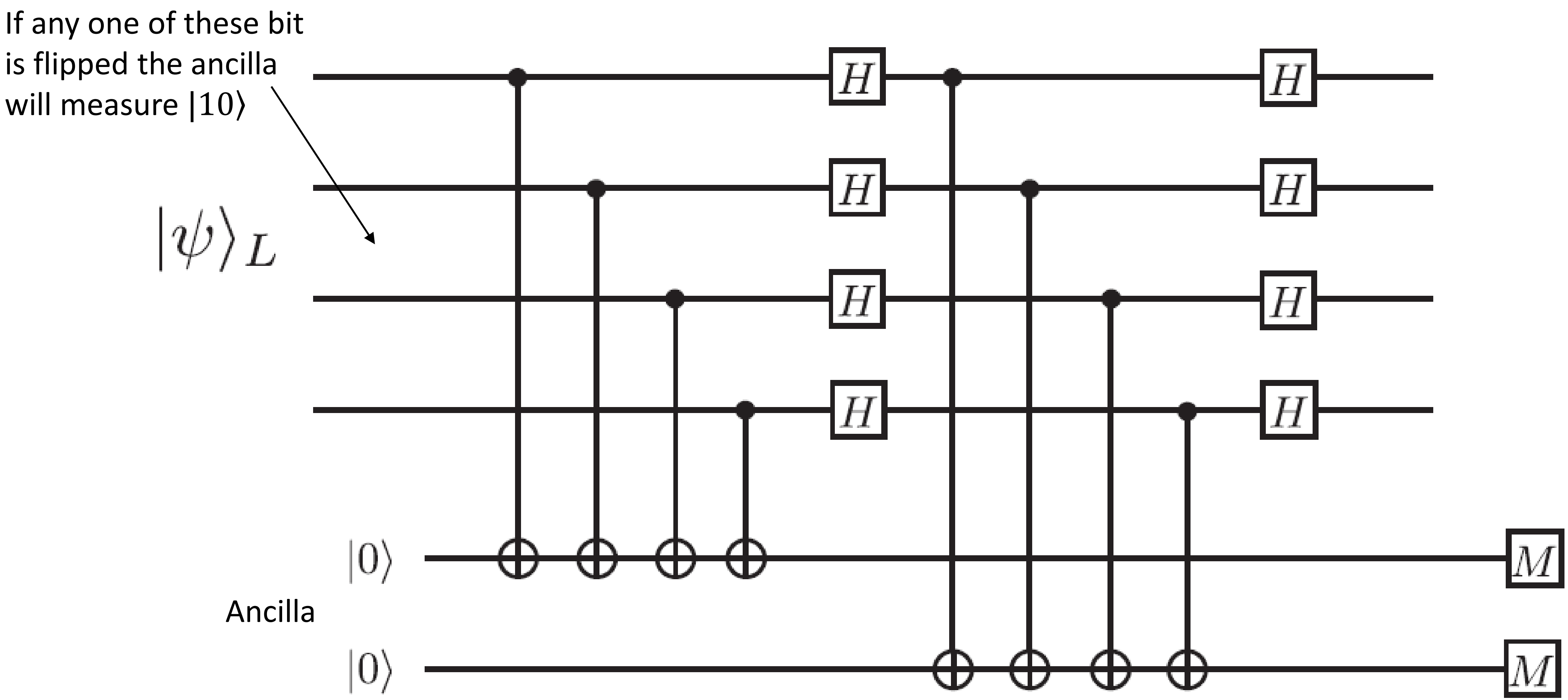


Complete Quantum Circuit for bit-flip



Bit-flip, Phase-flip Detection Circuit

(without knowing the location of error)



Bit-flip error √
Phase-flip error √

How to fix both bit-flip and phase-flip error in same circuit?

The Shore Code- The first full Quantum Code

(protect against **bit and phase flip** errors on a qubit)

First part of the circuit encodes the qubit using the three qubit phase code

Second part of the circuit encodes each of three bits using the bit flip code

The basis state for codes

Logical $|0\rangle$ and $|1\rangle$ states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

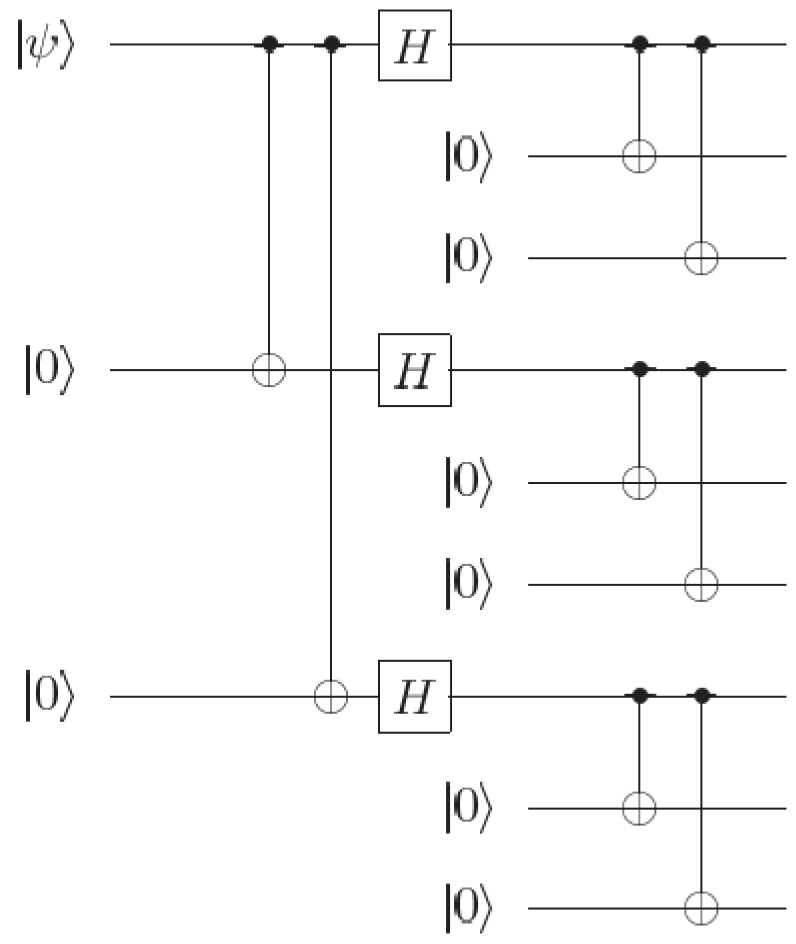
$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Bit flip error

Comparing Z_1Z_2 and Z_2Z_3 measurement we can find which of the bit is flipped (if any)

Same process we can continue to detect the error any of the nine quits

Encoding circuit for the Shore nine qubit code



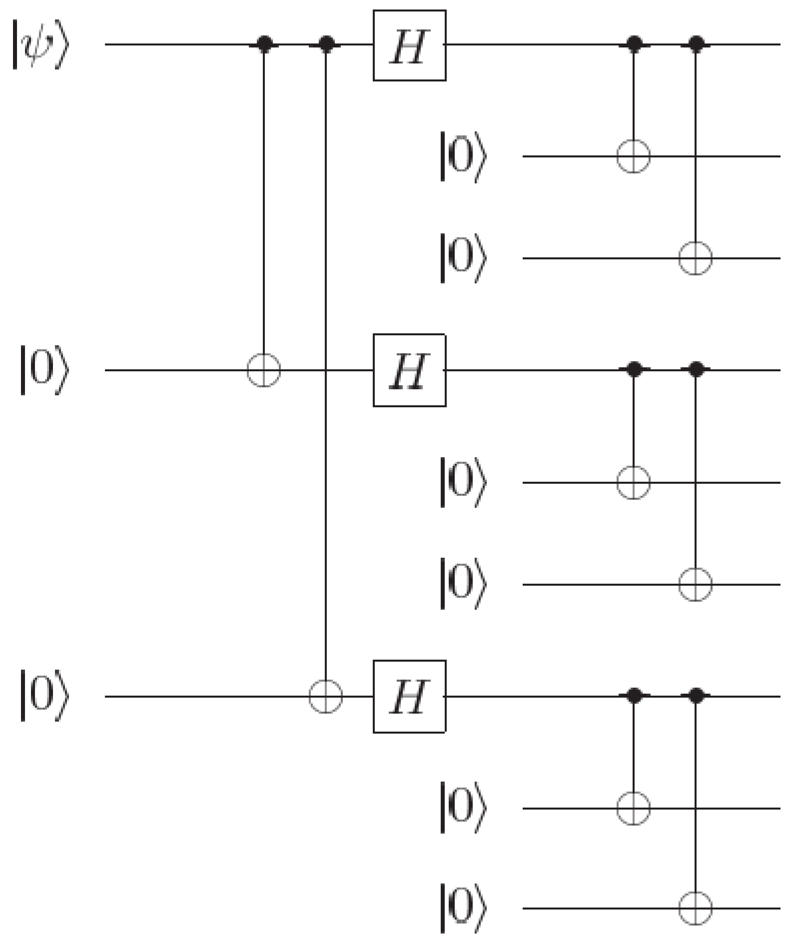
The Shore Code

(protect against bit and phase flip errors on any qubit)

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Encoding circuit for the Shore nine qubit code



Phase flip error

Suppose phase flip occurs in first qubit

Such phase flip flips **the sign of the first block of qubits**, i.e.

$|000\rangle + |111\rangle \longleftrightarrow |000\rangle - |111\rangle$

Phase flip on any of first three qubits has this effect

Phase flip syndrome measurements

Compare the **sign of 1st and 2nd blocks** of three qubits and then compare **the sign 2nd and 3rd blocks** of three qubits

The Shore Code

(protect against bit and phase flip errors on any qubit)

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Bit flip error in case of 9 bit Shor's code can be detected by the processes already described, i.e.

Compare Z_1Z_2 Measurement with Z_2Z_3

Compare Z_2Z_3 Measurement with Z_3Z_4

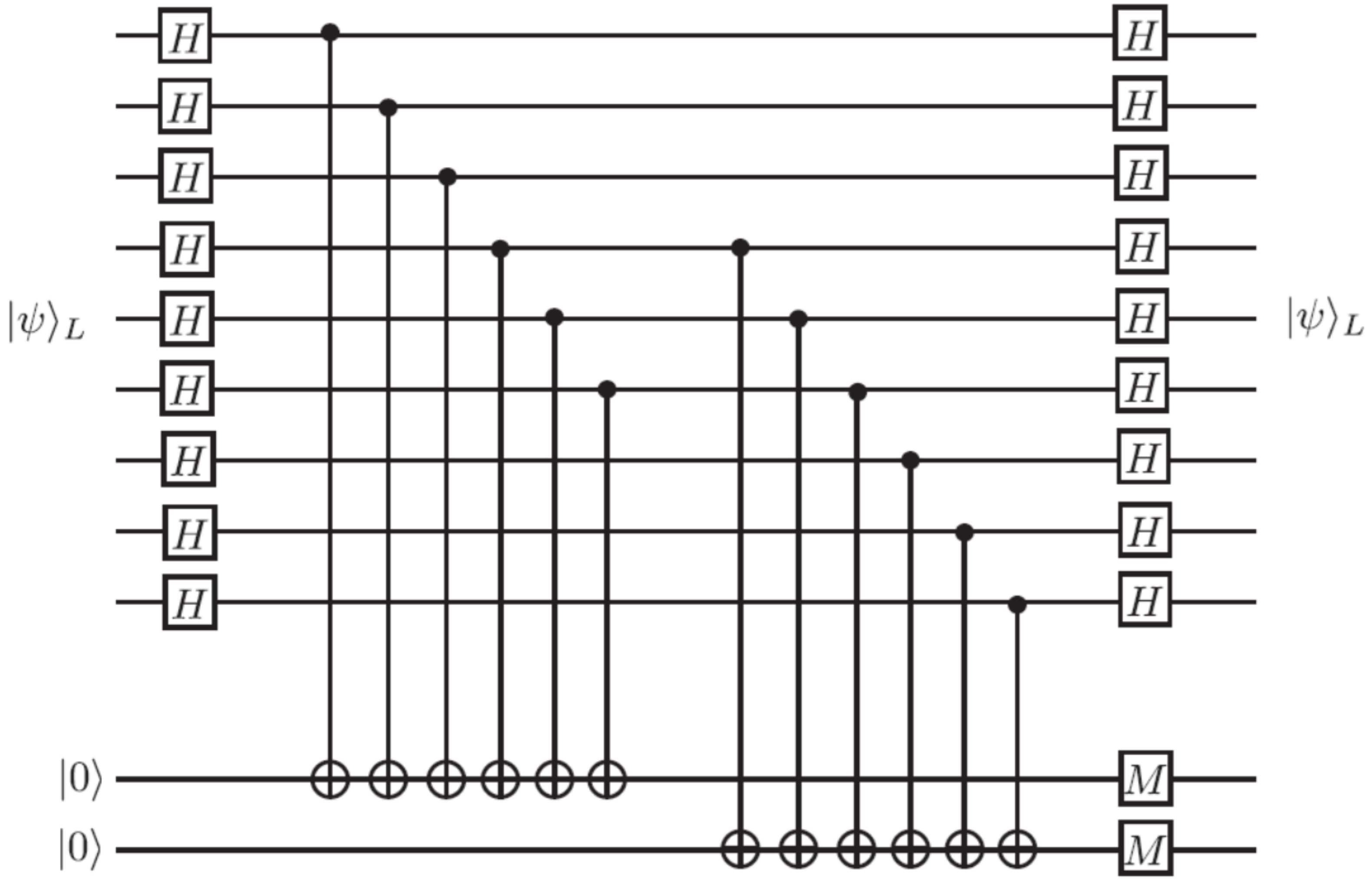
Equivalently Syndrome measurement for detecting phase flip errors

Phase flip in any of first three will have same phase error effect, therefore we have to 'compare the set of three' for phase flip

compare $X_1X_2X_3X_4X_5X_6$ Measurements with $X_4X_5X_6X_7X_8X_9$

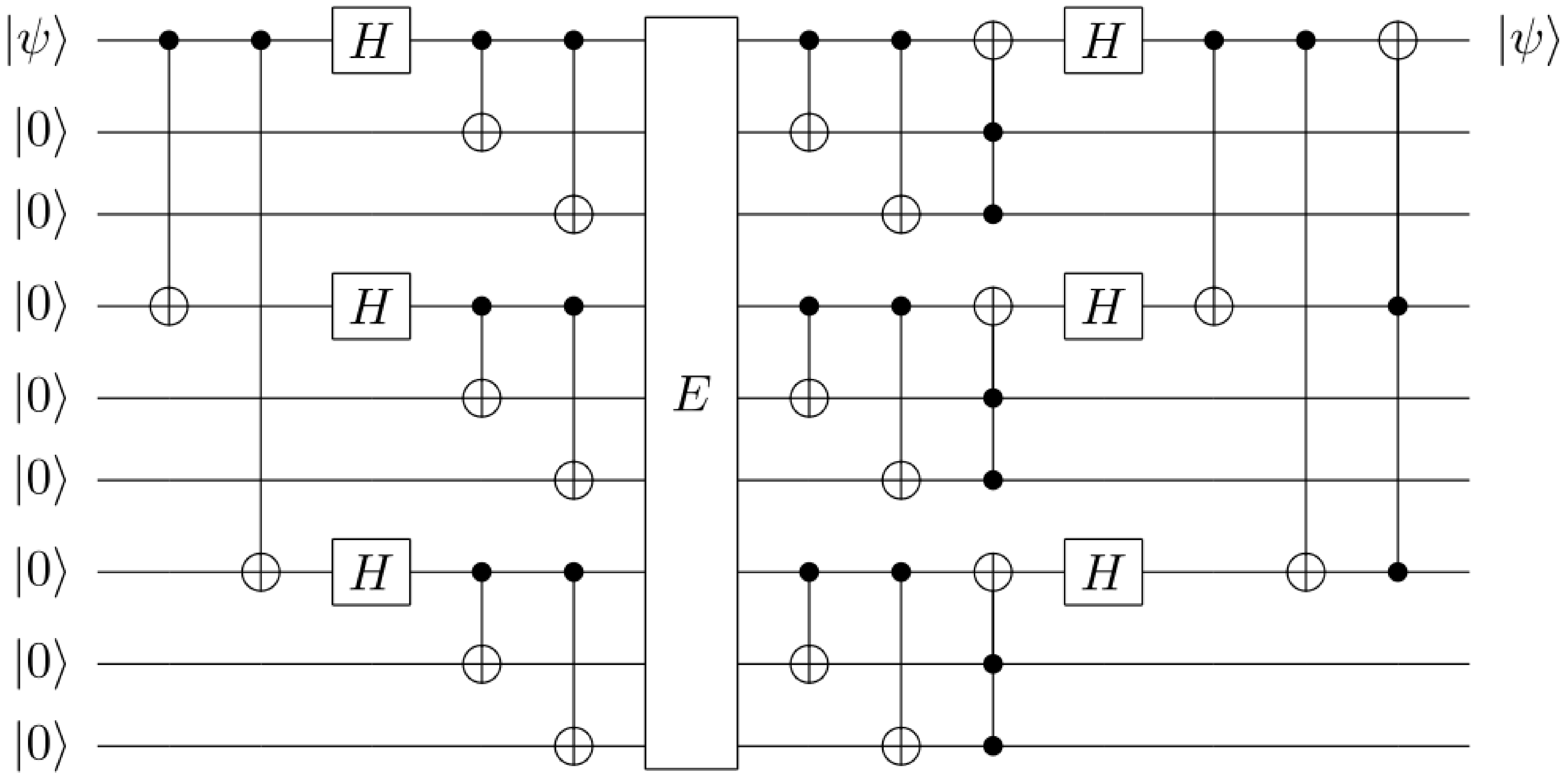
And recovery from phase flip on any of the first three qubits may be accomplished by applying the operator

Phase flip Error Correction Circuit



Circuit to perform Z-error correction for the 9-qbit code

Complete Shore Code Circuit



Properties of any Quantum Code

In order for code to correct two errors $E_a$and E_b, we must always be able to distinguish error E_a acting one basis codeword $|\psi_i\rangle$ from error E_b acting on a different basis codeword $|\psi_j\rangle$. We can only be sure of doing this if $E_a\ |\psi_i\rangle$ is orthogonal to $E_b\ |\psi_j\rangle$, i.e.

$$\langle\psi_i|E_j^\dagger E_b\ |\psi_j\rangle = 0$$

Stabilizer Formalism

The majority of error correcting codes that are used within literature are members of a class known as **stabilizer codes**

The general formalism applies broadly and there exits general rules to **construct preparation circuits**, **correction circuits** and **fault-tolerant logical gate operations** once the stabilizer of the code is known.

Describing quantum states in terms of operators rather than the states

A state $|\psi\rangle$ is defined to be **stabilized by some operator M** , if it is a +1 eigenstate of **M**

$$\boldsymbol{M}|\psi\rangle = |\psi\rangle$$

$$\hat{Z}|0\rangle = |0\rangle$$

$$\hat{X}|+\rangle = |+\rangle$$

Stabilizer for Different well Known States

State

Stabilizer

$$|\mathrm{GHZ}\rangle_3 = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

$$K_1 = \hat{X}\hat{X}\hat{X}$$
$$K_2 = \hat{Z}\hat{Z}\hat{I}$$
$$K_3 = \hat{I}\hat{Z}\hat{Z}$$

$$K_1|GHZ\rangle = |GHZ\rangle$$

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}},$$
$$|\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}},$$

$$\Phi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = ZZ \end{pmatrix} \quad \Phi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = ZZ \end{pmatrix}$$
$$\Psi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = -ZZ \end{pmatrix} \quad \Psi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = -ZZ \end{pmatrix}$$

$$K^1|\Phi^+\rangle = |\Phi^+\rangle$$

Stabilizer for Shor's nine-qubit Code

Logical state of a qubit in 9-bit Shor's code

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Logical state in Shor's code is eigenvectors of all eight (M_1 –M_8) operators with eigen value +1

	M_1	Z	Z	I	I	I	I	I	I	I
	M_2	Z	I	Z	I	I	I	I	I	I
Bit flip error	M_3	I	I	I	Z	Z	I	I	I	I
	M_4	I	I	I	Z	I	Z	I	I	I
	M_5	I	I	I	I	I	I	Z	Z	I
	M_6	I	I	I	I	I	I	Z	I	Z
Phase flip error	M_7	X	X	X	X	X	X	I	I	I
	M_8	X	X	X	I	I	I	X	X	X

7 Qubit Stabilizer

7 qubit logical state

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle),$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle).$$

Stabilizer set for 7 qubit code

$$K^1 = IIIXXXX, \quad K^2 = XIXIXIX,$$
$$K^3 = IXXIIXX, \quad K^4 = IIIZZZZ$$
$$K^5 = ZIZIZIZ, \quad K^6 = IZZIIZZ.$$

Quantum Error Correction with Stabilizer Code

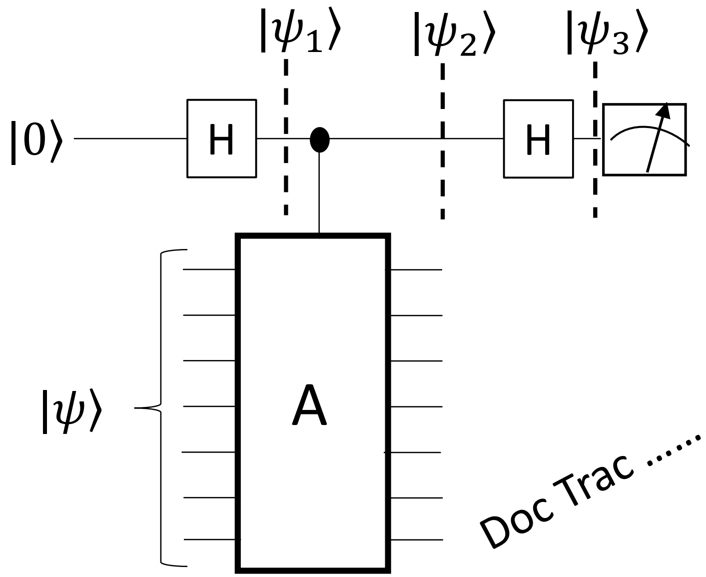
The use of stabilizer formalism to **describe quantum error correction codes** is extremely useful since it allows easy synthesis of correction circuits and also allows for quick determination of what logical operations can be applied directly on encoded data. The link between **stabilizer codes** and **stabilizer states** comes about by defining a relevant coding *subspace* within the larger Hilbert space of a multi-qubit system

Analog Measurements are Projective Measurements

Projector in +1 eigen state of A $\equiv P_{+}^{A} = \frac{\hat{I}+\hat{A}}{2}$ $\qquad P_{+}^{A}|\psi\rangle = \frac{\hat{I}+\hat{A}}{2}|\psi\rangle$ $\qquad P_{+}^{A}|\psi\rangle = \frac{1+1}{2}|\psi\rangle$

Projector in -1 eigen state of A $\equiv P_{-}^{A} = \frac{\hat{I}-\hat{A}}{2}$ $\qquad P_{-}^{A}|\psi\rangle = \frac{\hat{I}-\hat{A}}{2}|\psi\rangle$ $\qquad P_{-}^{A}|\psi\rangle = \frac{1-1}{2}|\psi\rangle$

IZZ and ZZI are 'three bit code' stabilizer



Doc Trac ……. Error correction scheme

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)$$

$$|\psi_2\rangle = CA\left\{\frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)\right\}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle A|\psi\rangle)$$

$$|\psi_3\rangle = \frac{1}{2}[(|0\rangle + |1\rangle)|\psi\rangle + (|0\rangle - |1\rangle)A|\psi\rangle)]$$

$$|\psi_3\rangle = \left(\frac{|0\rangle + |1\rangle}{2}\right)|\psi\rangle + \left(\frac{|0\rangle - |1\rangle}{2}\right)A|\psi\rangle$$

$$|\psi_3\rangle = |0\rangle\left(\frac{\hat{I}+\hat{A}}{2}\right)|\psi\rangle + |1\rangle\left(\frac{\hat{I}-\hat{A}}{2}\right)|\psi\rangle$$

$$|\psi_3\rangle = |0\rangle(P_{+}^{A})|\psi\rangle + |1\rangle(P_{-}^{A})|\psi\rangle$$

Measured output is either +1 eigen state or -1 eigen state of projector

Error correction Code

- Repetition code (majority voting)
- Surface Code
- Bosonic code

For d repetition

$$P = \sum_{n=0}^{|d/2|} \binom{d}{n} p^n (1-p)^n \sim \left(\frac{p}{1-p}\right)^{|d/2|}$$

P decays exponentially with d

Techniques

Syndrome Measurements
Decoding
Logical Operations

Quantum error correcting codes are defined by the measurements we make
Let us move beyond the simple ZjZj+1 of the repetition of the code
In the surface code we use 2D lattice of code qubits, and define observables for plaquettes and vertices

Error correction tools developed theoretically, like ‘working on algorithms, although it depends upon hardware

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Stabilizer

Consider the two qubit Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\hat{X}_1\hat{X}_2|\psi\rangle = |\psi\rangle$ $\hat{Z}_1\hat{Z}_2|\psi\rangle = |\psi\rangle$

The state $|\psi\rangle$ is stabilized by the operator $\hat{X}_1\hat{X}_2$ and $\hat{Z}_1\hat{Z}_2$

For n qubits, n commuting independent Pauli operators stabilizes one state

$S = \langle ZII, IZI, IIZ\rangle \rightarrow |000\rangle$

Measuring $\hat{Z}\otimes\hat{Z}$ detects bit flip errors

$S = \langle XII, IZI, IIY\rangle \rightarrow |+0-i\rangle$

Measuring $\hat{X}\otimes\hat{X}$ detects phase flip errors

$S = \langle XX, ZZI, IZZ\rangle \rightarrow |???\rangle$

Error syndrome is formed by measuring enough operators to determine the location of error

Quantum Error Detection

Up to now we discussed the **error detection and correction processes**, however in another approach of QEC, only error detection play an important role without enforcing the error correction

The simplest error detecting circuit is the 4-qubit code. This encodes two logical qubits on to four physical qubits with the ability to detect a single error on either of the two logical qubits

Four basis states for the codes are

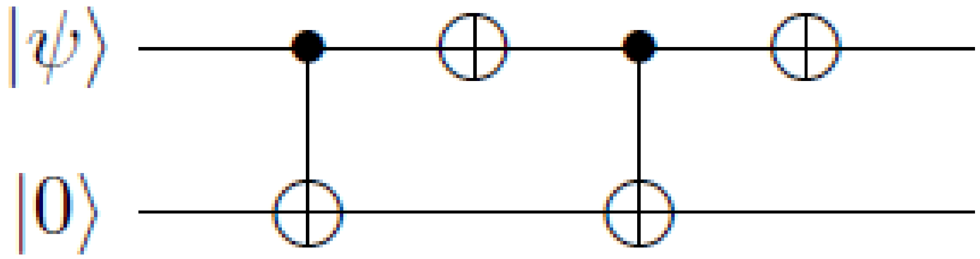
$$|00\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle),$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle),$$
$$|11\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle).$$

Leakage Error

Qubit is neither in state $|0\rangle$ $nor\ in\ \ state$ $|1\rangle$, information is leaked, liked qubit in state $|3\rangle$, $|4\rangle$, etc. Fock state

Assuming $|\psi\rangle$ $is\ in$ $|0\rangle$ or $|1\rangle$ state, if $|\psi\rangle$ is in state $|0\rangle$ or $|1\rangle$ **ancilla bit will flip** from $|0\rangle$ to $|1\rangle$.

But if $|\psi\rangle$ is NOT in state $|0\rangle$ or $|1\rangle$, ancilla will reman in $|0\rangle$ state



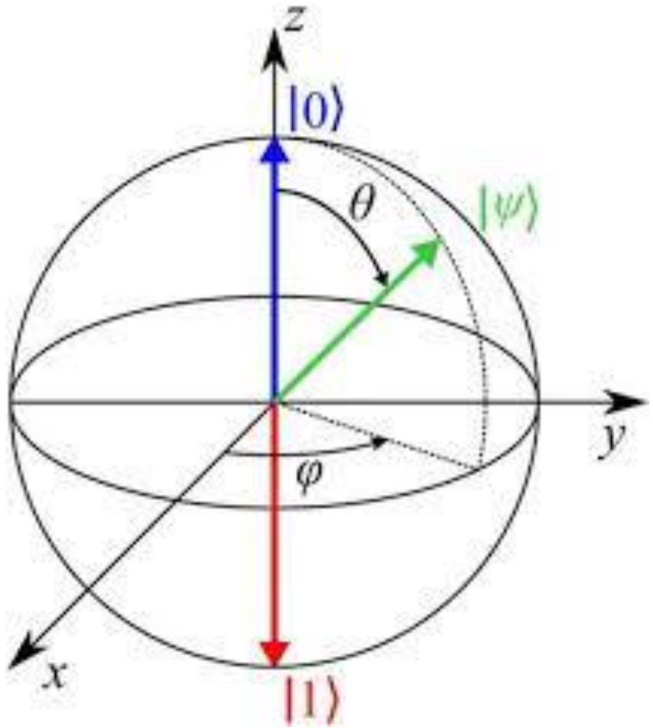
Circuit to detect the leakage error

The need of error correction

Quantum computers have a great deal of potential, but to realize that potential, they need some sort of protection from noise.

Classical computers are digital: after each step, they correct themselves to the closer of 0 or 1 (parity check).

Quantum computers have a continuum of states, so it would seem, at first glance, '**quantum error can not be corrected**'.



For instance, a likely source of error is over-rotation:

a state $\alpha|0\rangle + \beta|1\rangle$ might be supposed to become $\alpha|0\rangle + \beta e^{i\varphi}|1\rangle$, but instead becomes $\alpha|0\rangle + \beta e^{i(\varphi+\delta)}|1\rangle$.

The actual state is very close to the correct state, but it is still wrong.

If we don't do something about this, the **small errors will build up over the course of the computation, and eventually will become a big error.**

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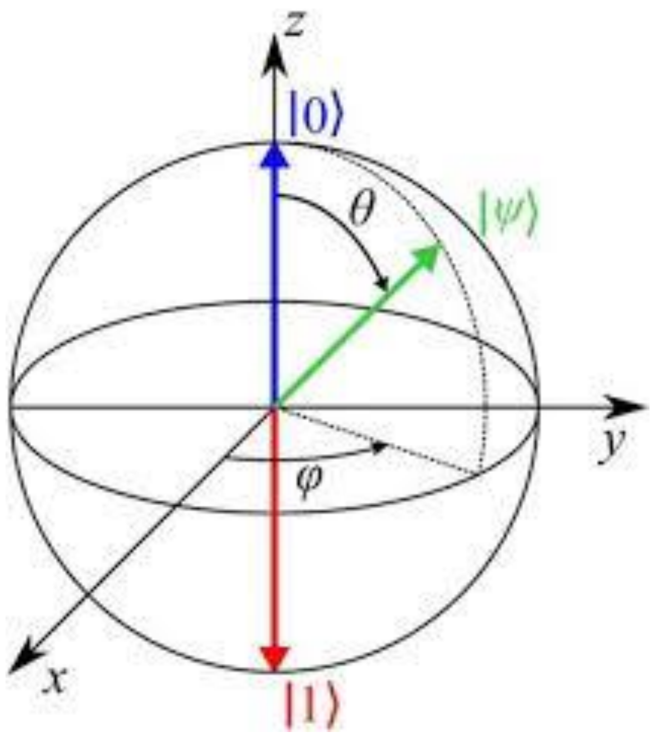

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