PHYS 512

Week 05 and 06

Review of Linear Algebra

(Reference to Quantum Computing)

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Linear Algebra

You can review these topics from

David McMahon – Ch 2, Ch 3 and Ch 4

Bernard Zygelman – Ch1 and Ch 2

Complex Vector Space

The vector spaces encountered in physics are mostly real vector spaces and complex vector spaces. Classical mechanics and electrodynamics are formulated mainly in real vector spaces **while quantum mechanics (and hence this course) is founded on complex vector spaces.**

In the rest of this chapter, we briefly summarize vector spaces and matrices (linear maps), taking applications to quantum mechanics into account.

Outline

- Review of basics of linear algebra
- Concept of State or Vector or Ket
- Vector Space and Hilbert space
- Linear Combination of Vectors
- Linear Independence
- Uniqueness of spanning Space
- Basis and dimension
- Inner Produce
- Outer Product

Linear Algebra

- Quantum Theory is based on the construct: wave function and operators

- The **state** of a system is represented by its **wave function**, **observables** are represented by **operator**

- Wave functions satisfy the defining conditions for abstract 'vectors' and operators act on them as 'linear transformation'

- Therefore, **Linear algebra** is the language of quantum computing

- The goal of this section is to create a foundation of introductory linear algebra knowledge,

- upon which the reader can build during their study of quantum computing.

Probability Basics

Probability heavily used is quantum theory to predict the possible results of measurements

Probability p_i of an event x_i falls in the range $0 \leq p_i \leq 1$

0 – impossible
1 – certain to happen

The probability of an event is simply the **relative frequency of its occurrence**

Suppose there are n total events, the *jth* event occur n_j times, and we have $\sum_{j=1}^{\infty} n_j = n$,
then the probability that the *jth* even occur

$$\boxed{p_i = \frac{n_j}{n}}$$

The sum of all the probabilities is 1

$$\boxed{\sum_{j=1}^{\infty} p_j = \sum_{j=1}^{\infty} \frac{n_j}{n} = \frac{1}{n} \sum_{j=1}^{\infty} n_j = \frac{n}{n} = 1}$$

The **average value** of the distribution is referred to as the **expectation value in quantum mechanics**, given by

$$\langle j \rangle = \sum_{j=1}^{\infty} \frac{j n_j}{n} = \sum_{j=1}^{\infty} j p_j$$

and **variance of distribution** is
(Standard deviation)

$$\boxed{\langle (\Delta j)^2 \rangle = \langle j^2 \rangle - \langle j \rangle^2}$$

Quantum Mechanics Toolbox

Bits and Qubits

- Basic unit of information is called a bit (like Yes/No, on/off, stop/go etc.)
- All digital computing machines are constructed from individual bits
- Integer value 0 and 1 denote the value of a bit
- The **qubit** is a similar but **distinct concept**

Possible states of a classical bit

$$0 \rightarrow |0\rangle$$

$$1 \rightarrow |1\rangle$$

Possible states of a **qubit**

Basic Principals of Quantum Mechanics

- According to the principles of quantum mechanics, systems are set to a definite state only once they are measured.

- Before a measurement, systems are in an indeterminate state; after we measure them, they are in a definite state

- If we have a system that can take on one of two discrete states when measured, we can represent the two states in **Dirac notation** as $|0\rangle$ and $|1\rangle$

- We can then represent a superposition of states as a linear combination of these states, such as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where $\alpha^2 + \beta^2 = 1$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

α and β generally complex numbers

with definite values of α and β

Postulates of Quantum Mechanics

Before we deal with 'exact postulates of QM' let us consider them in reference to array of five lamps

Postulate - I

QM - Following a measurement (observation) we observe only one of out of 32 possible ON/OFF configuration immediately after the measurements. CM – measurements will give you the same state

Looks obvious!

Postulate – II a

32 possible states of array of 5 lamps are vectors in a linear vector space

According to this postulate and '+' of linear vector space, $|11010\rangle + |00101\rangle$ is also a vector in this space, **a possible state!**

$$|\Phi\rangle \equiv |00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle + |00110\rangle + |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle + |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle + |10101\rangle + |10110\rangle + |10111\rangle + |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle + |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle,$$

System do exist in linear combination of theses states!

But **can not be observed/measured**

Upon measurement/observation system will **collapse** into 'one of these 'states

Which and Why ?

The quantum state $|\psi\rangle$, expressed as a linear combination of other states, is sometime called ***superposition principle***

Postulate – II b

A complete physical description of this quantum register is encapsulated by a vector $|\psi\rangle$ in this vector space.

Hilbert Space

Hilbert Space is a vector space

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space **to spaces with any finite or infinite number** of dimensions.

Linearly Independent vectors

Like Unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in 2-D or 3-D, are they linearly independent?

We pointed out that if $|\alpha\rangle$ is a vector so is $c\,|\alpha\rangle$ where c is a scalar quantity.
In Hilbert space the scalar c is generally a complex number.

Consider linear combination of n vector

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots c_n|\alpha_n\rangle$$

If sum equals to the **null vector 0** ONLY if $c_1 = c_2 = c_3 \ldots \ldots \ldots \ldots \ldots = c_n = 0$

Then set of vectors $\boxed{|\alpha_1\rangle, |\alpha_2\rangle, |\alpha_3\rangle \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots |\alpha_n\rangle}$ are linearly independent

A space that admits *n* linear independent vectors, but not *n*+1, is called an *n*-dimensional space

Nothing new !
Except the # of dimensions
(at least up to now)

In general, a Hilbert space allows infinite dimension but we are primarily concerned, in this text, with Hilbert spaces spanned by a finite and denumerable set of basis vectors.

Can you write

$\boxed{\hat{\imath} \text{ in terms of } \hat{\jmath} \;\&\; \hat{k}}$ NO

Hilbert space of array of five lamps has 32 dimension; **32 independent possible states**

Quantum Mechanics Toolbox

Binary Arithmetic

Imagine if you have a set of lamps, how many ways you can have ON/OFF arrangements of lamps

Depends upon number of lamps

How many possibilities for 5 and 7 set of lamps ?

Binary of number '26' - 11010

Possible combinations of n-bits – 2^n

If we have array of five lamps, in ON/OFF state, say ON, ON, OFF, ON,OFF

In QM we write 'this state' of set/register of lamps it as $|11010\rangle$ One of 32 ($=2^5$) possible states

Binary of 26

States can be added? like binary numbers $|11010\rangle + |00101\rangle$ $= |11111\rangle$

Two arrangements of ON/OFF

All ON

Does NOT make any sense!

Array of lamps in two different states at the same time

Is it possible ?

Quantum Mechanics Toolbox

Binary Arithmetic

- **Numbers in base 2** (1101, 01100111, 1011000111, …….)
- **Numbers in base 10** (5, 67, 893, …….)
- Convert the binary number 110011101 to a base 10 number

Binary numbers can be added or subtracted like base 10 numbers

Possible combinations of n-bits – 2^n

Possible states (combination) of a two qubits system: $2^2 = 4$

$|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

Possible states of a three qubits system: $2^3 = 8$

$|000\rangle$ $|001\rangle$ $|010\rangle$ $|011\rangle$ $|100\rangle$ $|101\rangle$ $|110\rangle$ $|111\rangle$

Similarly you can write possible states of n= 4, 5, 6 and so on number of qubit system

Linear Vector Space

Mathematically and conceptually very different from 2-D and 3-D 'Real' vector

Consider a set of objects $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots$.. We say that these objects belong to a ***linear vector space V*** provided that

(i) There exists an operation, which we denote by the **+ sign**, so that if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are any two members of the vector space *V* then so is the quantity $\boldsymbol{\alpha} + \boldsymbol{\beta}$.

(ii) For scalar *c*, there exists a scalar multiplication operation defined so that if *β* is a vector in *V* then so is $c\boldsymbol{\beta} = \boldsymbol{\beta}c$. If *a, b* are scalars $a\, b\, \boldsymbol{\beta} = a(b\, \boldsymbol{\beta})$.

(iii) Scalar multiplication is distributive, i.e. $c(\boldsymbol{\alpha} + \boldsymbol{\beta}) = c\,\boldsymbol{\alpha} + c\,\boldsymbol{\beta}$, also for scalar *a, b*, $(a + b)\alpha = a\alpha + b\alpha$.

(iv) The + operation is associative, i.e. $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma}) = (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma}$.

(v) The + operation is commutative, i.e. $\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\beta} + \boldsymbol{\alpha}$.

(vi) There exists a null vector 0 which has the property $0 + \alpha = \alpha$ for very vector α in *V* .

(vii) For every α in *V* there exists an inverse vector $-\alpha$ that has the property $\alpha + (-\alpha) = 0$.

Read these postulates, if you have question, please discuss

State Vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where $\alpha^2 + \beta^2 = 1$

$$\alpha^2 = |\alpha|^2 = (\alpha)(\alpha^*)$$

$$\beta^2 = |\beta|^2 = (\beta)(\beta^*)$$

α and β are called **probability amplitudes**; **NOT probabilities**

Generally α and β are complex numbers and they can be negative as well !

Probability of finding system in state $|0\rangle$ $is\ P_{|0\rangle} = \alpha^2$ and in state $|1\rangle$ $P_{|1\rangle} = \beta^2$

Generally speaking, if an event has N possible outcome and label the probability of finding 'i ' by Pi, then

$$\sum_{i=1}^{N} P_i = P_1 + P_2 + \cdots P_N = 1$$

That is why $\alpha^2 + \beta^2 = 1$

Remember

For a complex number z

$$z = x + iy$$

$$z^2 = |z|^2 = (z)(z^*)$$

$$z^2 = (x + iy)(x - iy)$$

$$z^2 = (x)^2 - (-iy)^2$$

$$z^2 = x^2 + y^2$$

Hilbert Space (Vector Space)

Basis and Dimensions

When a set of **vectors is linearly independent** and they span a space, the set is known a 'basis'

Like $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \widehat{\boldsymbol{k}}$ in 3-D Euclidean space

State vector in 2 D vector space

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

States in Matrix form

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In matrix form

$$|\Psi\rangle = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

State vector in n-D vector space

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \\ \\ a_n \end{pmatrix}$$

Example

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$$

In matrix form

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

Different Basis Set

$|0\rangle$ and $|1\rangle$ is common basis set

$|0\rangle$ and $|1\rangle$ are orthogonal to each other $\langle 0|1\rangle = 0$

$|+\rangle$ and $|-\rangle$ is another common set of basis

$|+\rangle$ and $|-\rangle$ are also orthogonal to each other $\langle +|-\rangle = 0$

Problem -01

Prove that $\langle +|-\rangle = 0$ $\langle +|+\rangle = 1$

Orthonormality of $|0\rangle$ and $|1\rangle$

Evaluate $\langle +|0\rangle =?$

Solve it yourself !

Problem -02

In $|0\rangle$, $|1\rangle$ basis a quantum state $|\psi\rangle$ is (matrix form)

$$|\psi\rangle = \begin{pmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{pmatrix}$$

(a) Express $|\psi\rangle$ in ket notation in $|0\rangle$, $|1\rangle$ basis

(b) Express $|\psi\rangle$ in ket notation in $|+\rangle$, $|-\rangle$ basis

Solve it yourself !

Hilbert Space (Vector Space)

A spanning set of vectors for a given space 'V' is **not unique**

Multiple basis set can be used!

Like set of orthogonal axis are not unique in 2D and 3D Euclidean space

State vector in 2 D vector space

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad and \quad |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

Spans $\mathbb{C}^2$, the vector space in which qubit lives

In matrix form $$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Now consider **new basis**

$$|U_1\rangle = |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad and \quad |U_2\rangle = |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

This set also spans the space $\mathbb{C}^2$; $\Rightarrow |U_1\rangle$ and $|U_2\rangle$ can be used to represent a state $|\psi\rangle$ of qubit

Consider $$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow |+\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\Rightarrow |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow \boxed{|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$$

Similarly show that

$$\boxed{|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}$$

Dirac's Ket and Bra Notations

The ***inner, or scalar product*** is a Hilbert space structure that provides a measure of the **degree of "overlap" between two vectors**.

Like dot-product in 2/3-D Euclidian space

Remember Hilbert space can be of infinite dimension - fairly abstract

Dot-product concept is useful to grasp the basic concept **of inner product** BUT DO NOT Stick to it

Vectors in Hilbert space represented by $|\psi\rangle$ Called 'ket' notation

Vector in Dual Space
(like mirror image of vector in Hilbert space) $\langle\psi|$ Called 'bra' notation

Every **ket** has **bra** counterpart in a space called '**dual space**"

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle \ldots\ldots\ldots\ldots\ldots c_n|\alpha_n\rangle \xrightarrow{\text{Ket to bra}} \langle\psi| = c_1^*\langle\alpha_1| + c_2^*\langle\alpha_2| + c_3^*\langle\alpha_3| \ldots\ldots\ldots. c_n^*\langle\alpha_n|$$

Expansion coefficients are *complex conjugate*

$\cancel{|\Phi\rangle + \langle\psi|}$ Ket and bra can not be added

$\langle\Phi|\psi\rangle$ Called inner product – to evaluate **complex number**

$|\Phi\rangle\langle\psi|$ Called outer product – not a scaler or vector – **an operator**

Dirac's Ket and Bra Notations

Inner Product

$\langle\Phi|\psi\rangle$ Called inner product – to evaluate **complex number**

Inner product provides a **measure of overlap** between the vector $|\Phi\rangle$ and $|\psi\rangle$

$\langle\Phi|\psi\rangle = \langle\psi|\Phi\rangle^*$ They are complex conjugate of each other

$\langle\psi|\psi\rangle = \langle\psi|\psi\rangle^*$ must be real and $\langle\psi|\psi\rangle \geq 0$

$\sqrt{\langle\psi|\psi\rangle} = || \, |\psi\rangle ||$ as ***length of a vector****, also called* ***'norm'***

If $\langle\psi|\psi\rangle = 1$ vector is of unit length or ***normalized***

For given vectors

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle| \quad \text{and} \quad \langle\Phi| = d_1^*\langle\alpha_1| + d_2^*\langle\alpha_2|$$

The inner product

$$\langle\Phi|\psi\rangle = (|d_1^*\langle\alpha_1| + d_2^*\langle\alpha_2|)((c_1|\alpha_1\rangle + c_2|\alpha_2\rangle|)$$

$$\langle\Phi|\psi\rangle = c_1 d_1^*\langle\alpha_1|\alpha_1\rangle + c_1 d_2^*\langle\alpha_2|\alpha_1\rangle + c_2 d_1^*\langle\alpha_1|\alpha_2\rangle + c_2 d_2^*\langle\alpha_2|\alpha_2\rangle$$

Complete it using Orthogonality condition $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$

Problem -03

$$\boxed{|\psi\rangle = \frac{3i}{5}|0\rangle + \frac{4}{5}|1\rangle}$$

Is the state normalized?

$$\alpha^2 + \beta^2 = 1$$

$$\left(\frac{3i}{5}\right)\left(\frac{-3i}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) = \frac{9}{25} + \frac{16}{25} = 1$$

(a) What is the prob that the state $|\psi\rangle$ collapse to $|1\rangle$

$$P_{|1\rangle} = \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) = \frac{16}{25}$$

2nd option is to take the inner product of $|\psi\rangle$ with $|1\rangle$

$$P_{|1\rangle} = |\langle 1|\psi\rangle|^2$$

$$P_{|1\rangle} = \left|\frac{3i}{5}\langle 1|0\rangle + \frac{4}{5}\langle 1|1\rangle\right|^2$$

We know $\langle 1|0\rangle = 0$

$\langle 1|1\rangle = 1$

$$P_{|1\rangle} = \left|\frac{4}{5}\right|^2 = \frac{16}{25}$$

(b) What is the prob that the state $|\psi\rangle$ collapse to $|\varphi\rangle$

Where

$$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{(1-i)}{2}|1\rangle$$

$$P_{|\varphi\rangle} = |\langle\varphi|\psi\rangle|^2$$

$$P_{|\varphi\rangle} = \left|\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{(1-i)}{2}\langle 1|\right)\left(\frac{3i}{5}|0\rangle + \frac{4}{5}|1\rangle\right)\right|^2$$

$$P_{|\varphi\rangle} =$$

Complete it yourself!

Problem -04

For each of the following qubits, if a measurement is made, what is the probability that we find the qubit in state $|0\rangle$? What is the probability that we find the qubit in the state $|1\rangle$?

(a) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

(b) $|\phi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

(c) $|\chi\rangle = \frac{(1+i)}{\sqrt{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle$

(d) What is the probability that the state $|\psi\rangle$ collapse into state $|\chi\rangle$

Solve it yourself!

Dirac's Ket and Bra Notations

Few more rules

Quantum Mechanics is 'Linear'

$$\langle u|(|\alpha v + \beta w\rangle) = \alpha\langle u|v\rangle + \beta\langle u|w\rangle$$

$$(\langle \alpha u + \beta v)|w\rangle = \alpha^*\langle u|w\rangle + \beta^*\langle v|w\rangle$$

$$(|u\rangle)^\dagger = \langle u|$$

Dirac's Ket and Bra Notations

Postulate – III a

32 vectors (all possible states of array of five lamps) **form a basis** **that spans the Hilbert space of the array of five lamps**

$|\Phi\rangle \equiv$
$|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle + |00110\rangle$
$+ |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle$
$+ |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle$
$+ |10101\rangle + |10110\rangle + |10111\rangle + |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle$
$+ |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle,$

According to this postulate **any of these vectors** (states), for example $|11011\rangle$, **is orthogonal to all other vectors** (states) in set of 32. Basis vector are orthonormal

Orthogonality is a very important concept, mathematically and in physical systems

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle \dots\dots\dots\dots\dots c_n|\alpha_n\rangle$$

Ket and bra in summation form

$$|\psi\rangle = \sum_{i=1}^{n} c_i|\alpha_i\rangle$$

$$\langle\psi| = \sum_{j=1}^{n} c_j^* \langle\alpha_i|$$

If $|\psi\rangle$ has unit length i.e. $\langle\psi|\psi\rangle = 1$

$$\langle\psi|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_j^* c_i \left\langle\alpha_j\,|\alpha_i\right\rangle \xrightarrow{\text{Orthonormality}} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_j^* c_j \delta_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} |c_i|^2 = 1$$

Remember $|c_i|^2$ is probability of finding $|\psi\rangle$ and state $|\alpha_i\rangle$

Sum of all probabilities = 1

Dirac's Ket and Bra Notations

More about inner product

Taking inner product of $|\alpha_m\rangle$ with $|\psi\rangle$

$$\langle\alpha_m|\psi\rangle = \sum_{j=1}^{n} c_j\langle\alpha_m|\alpha_j\rangle \qquad = \sum_{j=1}^{n} c_j\delta_{mj} = c_m$$

Using Orthonormality

Before you process please check above in 3-D space! **For more intuitive understanding**

Remember inner product is like dot product in 2-/3-D

$$\hat{\imath}.\vec{A} = \hat{\imath}.(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}) = A_1$$

.........similarly, the Hilbert space vector

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle \dots \dots \dots \dots \dots c_n|\alpha_n\rangle$$

is a linear combination of **orthonormal basis** vector $|\alpha_i\rangle$

Using orthomormality we can find c_i

$$c_1 = \langle\alpha_1|\psi\rangle \qquad c_2 = \langle\alpha_2|\psi\rangle \qquad \dots \dots \dots \dots . c_n = \langle\alpha_n|\psi\rangle$$

Consider a vector in 2-/3-D

$$\boxed{\vec{A} = A_1\hat{\imath} + iA_2\hat{\jmath} + A_3\hat{k}_1}$$

To find A_1 , take dot product of A with $\hat{\imath}$

Dirac's Ket and Bra Notations
Outer Product

Dyadic in Euclidean space – a bilinear expression such as $\hat{\imath}\hat{\jmath}$ (or any one of other eight combinations)

Dyadic can be positioned before or after a vector $\hat{\imath}\hat{\jmath}\vec{A} = \hat{\imath}(\hat{\jmath}.\vec{A}) = \hat{\imath}A_y$ $\vec{A}\hat{\imath}\hat{\jmath} = (\vec{A}.\hat{\imath})\hat{\jmath} = \hat{\jmath}A_x$

Table 1.1 Examples and comparison of three-dimensional Euclidean space structures with n-dimensional Hilbert space analogues

Structure	Euclidean space	Hilbert space
Basis expansion	$\vec{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$	$\lvert\Psi\rangle = c_1\lvert\alpha_1\rangle + c_2\lvert\alpha_2\rangle + \cdots c_n\lvert\alpha_n\rangle$
Inner product	$\vec{A}\cdot\vec{B}$	$\langle\Phi\lvert\Psi\rangle$
Basis components	$A_x = \vec{A}\cdot\hat{\mathbf{i}}, \ldots$etc.	$c_i = \langle\alpha_i\lvert\Psi\rangle \quad i = 1, 2\ldots n$
Outer product	(dyadic) $\hat{\mathbf{j}}\,\hat{\mathbf{k}}, \ldots$etc.	$\lvert\alpha_i\rangle\langle\alpha_j\rvert$

Remember this table is only for intuitive purpose , Actually Hilbert space is different from Euclidean space!

Dirac's Ket and Bra Notations

Postulate – III b

Array of five lamps is in state $|\Phi\rangle$ than a measurement yields the lamps configuration corresponding to **one of 32 states** with probability

$|\Phi\rangle \equiv$
$|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle + |00110\rangle$
$+ |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle$
$+ |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle$
$+ |10101\rangle + |10110\rangle + |10111\rangle + |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle$
$+ |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle,$

$p_i = |c_i|^2 = |\langle i\, |\Phi\rangle|^2$ Where $|i\rangle$ is one of the 32 states of array of lamps

Where are $c_i's$

The condition $\langle\psi|\psi\rangle = 1$ insures that $\sum_i p_i = 1$

Consider a data of 1000 experiments fro array of five lamps

Table 1.2 Tally of results for the gedanken experiment

Measurement	00000	01000	10001	11000
# of observations	101	209	321	369

Therefore, if I know the probability (i.e. $|c_i|^2$) I can write expression for state vector $|\Phi\rangle$

$$|\psi\rangle = \sqrt{\frac{101}{1000}}|00000\rangle + \sqrt{\frac{209}{1000}}|01000\rangle + \sqrt{\frac{321}{1000}}|10001\rangle + \sqrt{\frac{369}{1000}}|11000\rangle$$

In Hilbert space '**expansion coefficients**' c_i's are complex numbers

Note: It is impossible to find the coefficients c_i's, **but $|c_i|^2$ can be**

If coefficient c_i is replaces by
(remember $|c_i|^2 = p_i$ prob. Measure)

$c_i \to c_i e^{i\beta}$ where β is called phase

Probability remain the same

State of Array of five lamps in Quantum Mechanics

$|\Phi\rangle \equiv$
$|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle$
$+ |00110\rangle + |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle$
$+ |01100\rangle + |01101\rangle + |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle$
$+ |10010\rangle + |10011\rangle + |10100\rangle + |10101\rangle + |10110\rangle + |10111\rangle$
$+ |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle + |11100\rangle + |11101\rangle$
$+ |11110\rangle + |11111\rangle$,

In QM notation

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + c_3|\alpha_3\rangle \ldots \ldots \ldots \ldots \ldots c_n|\alpha_n\rangle$$

Where $|\alpha_1\rangle$ is $|00000\rangle$ ……$|\alpha_{n=32}\rangle$ is $|11111\rangle$.

$|c_1|^2$ is probability of finding 'array' in state $|\alpha_1\rangle$

Quantum Mechanics is a probabilistic theory (Postulate III b) **where full knowledge of the system** i.e. $|\psi\rangle$, **does not guarantee a definite outcome for a measurement**

On measurements system collapse to a state, and remains in that state. Any consequent measurements will measure the system in same (previous) state

If we **DO NOT measure** (look at lamps) the system (array of 5 lamps), system can be in any of 32 possible states, however, once you measure (look at lamps) lamps will remain in that state. (If you DONOT look at the Moon, Moon is not there!!!!!)

Direct and Kronecker Product

To explore the Hilbert space in general we need to introduce the direct or tensor product.

We build higher-dimensional Hilbert space from direct products of single qubit Hilbert spaces

Remember single qubit belongs to 2-D Hilbert space

The ket $|0\rangle$ and $|1\rangle$ are possible basis vectors as we require them to be linearly independent and orthonormal.

In this Hilbert space (of a qubit, with bases vector $|0\rangle$ and $|1\rangle$) any state vector $|\Psi\rangle$ can be expressed as linear of these basis

The Direct or Tensor product

Direct product of two kets $|a\rangle$ and $|b\rangle$ is given by $|a\rangle \otimes |b\rangle$

If both kets $|a\rangle$ and $|b\rangle$ are single qubit basis kets $|a\rangle \otimes |b\rangle \equiv \quad |0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle$

Further remember $|a\rangle \otimes |b\rangle \neq |b\rangle \otimes |a\rangle$

Theorem 1.3 *Given a Hilbert space of dimension d that is spanned by basis vectors* $|b\rangle$*, and the single qubit vector* $|a\rangle$ *for* $a \in 0, 1$, the direct product $|a\rangle \otimes |b\rangle$ for all a, b are basis vectors in a Hilbert space of dimension 2d. The dual vector for

Direct and Kronecker Product

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$c|a\rangle \otimes |b\rangle$ is $c^*\langle b| \otimes \langle a|$ For all values of a, b.

$(c|ab\rangle)$ $\qquad$ $c^*\langle ba|$

Above theorem allow **inner product** with all vectors in the direct product Hilbert space

Assume $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ are single qubit states

$$|\Psi\rangle = c_1|a\rangle \otimes |b\rangle + c_2|c\rangle \otimes |d\rangle \qquad |\Phi\rangle = d_1|a\rangle \otimes |b\rangle + d_2|c\rangle \otimes |d\rangle$$

$$\langle\Psi|\Phi\rangle = (c_1^*\langle b| \otimes \langle a| + c_2^*\langle d| \otimes \langle c|)(d_1|a\rangle \otimes |b\rangle + d_2|c\rangle \otimes |d\rangle)$$

$$\langle\Psi|\Phi\rangle = c_1^* d_1\langle b|b\rangle\langle a|a\rangle + c_1^* d_2\langle b|d\rangle\langle a|c\rangle + c_2^* d_1\langle c|a\rangle\langle d|b\rangle + c_2^* d_2\langle c|c\rangle\langle d|d\rangle$$

$$\langle\Psi|\Phi\rangle = c_1^* d_1 + c_2^* d_2$$ Using orthonormality condition

$$|\Psi\rangle = c_1|ab\rangle + c_2|cd\rangle$$

$$|\Phi\rangle = d_1|ab\rangle + d_2|cd\rangle$$

$$\langle\Psi|\Phi\rangle = (c_1^*\langle ba| + c_2^*\langle dc|)(d_1|ab\rangle + d_2|cd\rangle)$$

$$\langle\Psi|\Phi\rangle = c_1^* d_1 \langle ba|ab\rangle + c_1^* d_1\langle ba|cd\rangle + c_2^* d_1 \langle dc|ab\rangle + c_2^* d_1\langle dc|cd\rangle$$

$$\langle\Psi|\Phi\rangle = c_1^* d_1 \langle ba|ab\rangle + c_1^* d_1 \langle ba|cd\rangle + c_2^* d_1 \langle dc|ab\rangle + c_2^* d_1 \langle dc|cd\rangle$$

$$\langle\Psi|\Phi\rangle = c_1^* d_1 \overset{=1}{\cancel{\langle b|b\rangle}}\langle a|a\rangle + c_1^* d_1 \overset{=0}{\cancel{\langle b|d\rangle}}\langle a|c\rangle + c_2^* d_1 \overset{=0}{\cancel{\langle d|b\rangle}}\langle c|a\rangle + c_2^* d_1 \overset{=1}{\cancel{\langle d|d\rangle}}\langle c|c\rangle$$

Problem -05

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{2}}|111\rangle \qquad |\varphi\rangle = \left(\frac{1+i}{2}\right)|100\rangle + \left(\frac{1-i}{2}\right)|101\rangle$$

Find $\langle\varphi|\psi\rangle = ?$

Solve it yourself!

$\langle \ldots \ldots .4th, 3rd, 2nd, 1st|$ $\quad\quad\quad\quad$ $|1st, 2nd, 3rd, 4th \ldots \ldots \rangle$

Problem -06

$$\langle 11|00\rangle =?$$

$$\langle 1110|0011\rangle =?$$

$$\langle 1110|0111\rangle =?$$

Where $|0\rangle$ and $|1\rangle$ are orthonormal basis

Problem -07

Consider following two states
of five lamp system

$|\psi\rangle = |10101\rangle$

$|\varphi\rangle = |11111\rangle$

Are they orthogonal to each other?

Can five lamps collapse above two states at the same time?

$\langle\psi|\varphi\rangle = ?$

Using ket-bra notation

$\langle\psi|\varphi\rangle = ?$

Using matrix notation

$$|\varphi\rangle = |11111\rangle = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{32 row} \end{array}$$

$$\langle\psi|\varphi\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = 0$$

$$|\varphi\rangle = |10101\rangle = \begin{pmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow \text{15 row} \\ \\ \\ \end{array}$$

Outer Product and Operators

Dirac's bra-ket formalism to construct an **outer product**

Consider states $|\psi\rangle = c_1|\alpha_1\rangle + c_2\,|\alpha_2\rangle$ and $|\Phi\rangle = d_1|\alpha_1\rangle + d_2|\alpha_2\rangle$

Dirac's distributive axioms for outer products $|\psi\rangle\langle\Phi| = (c_1|\alpha_1\rangle + c_2|\alpha_2\rangle)\,(d_1^*\langle\alpha_1| + d_2^*\langle\alpha_2|$

$$|\psi\rangle\langle\Phi| = c_1 d_1^*|\alpha_1\rangle\langle\alpha_1| + c_1 d_2^*\,|\alpha_1\rangle\langle\alpha_2| + c_2 d_1^*|\alpha_2\rangle\langle\alpha_1| + c_2 d_2^*|\alpha_2\rangle\langle\alpha_2|$$

Consider the outer product $\mathbf{X} = |\Phi\rangle\langle\psi|$

$\mathbf{X}|\Gamma\rangle\ and\ \langle\Gamma|\boldsymbol{X}$ are valid operations

$|\Gamma\rangle\mathbf{X}\ and\ \mathbf{X}\langle\Gamma|$ are NOT valid operations

How **outer product operator** can act on state

$\mathbf{X}|\Gamma\rangle = (|\Phi\rangle\langle\psi|)|\Gamma\rangle = |\Phi\rangle(\langle\psi|\Gamma\rangle) = c|\Phi\rangle$ Where c = $(\langle\psi|\Gamma\rangle)$

$\langle\Gamma|\mathbf{X} = \langle\Gamma|(|\Phi\rangle\langle\psi|) = (\langle\Gamma|\Phi\rangle)\langle\psi| = \langle\psi|$d Where d = $(\langle\Gamma|\Phi\rangle)$

Remember generally 'c' and 'd'- the inner products are **complex numbers**

Outer product $\mathbf{X} = |\Phi\rangle\langle\psi|$ act on vector $|\Gamma\rangle$ and transform it to $c|\Phi\rangle$ in Hilbert space

Outer product $\mathbf{X} = |\Phi\rangle\langle\psi|$ act on vector $\langle\Gamma|$ and transform it to $\langle\psi|$d in dual space

Outer products are **'operators'** in Hilbert space

Operator can change the sate of quantum system

Outer Product and Operators

$\mathbf{X}|\Gamma\rangle = (|\Phi\rangle\langle\psi|)|\Gamma\rangle = |\Phi\rangle(\langle\psi|\Gamma\rangle) = c|\Phi\rangle$

When outer product (**X**) act on ket result is also a ket

$\langle\Gamma|\mathbf{X} = \langle\Gamma|(|\Phi\rangle\langle\psi|) = (\langle\Gamma|\Phi\rangle)\langle\psi| = \langle\psi|\mathrm{d}$

When outer product (**X**) act on bra result is also a bra

Operators are objects that maps vectors to other vectors in Hilbert space

Additional properties

Now according to $\mathbf{X}|\Gamma\rangle = (|\Phi\rangle\langle\psi|)|\Gamma\rangle = |\Phi\rangle(\langle\psi|\Gamma\rangle) = c|\Phi\rangle$

The **dual of transformed vector** of $\mathbf{X}|\Gamma\rangle$ is $\langle\Phi|c^*$ where $c^* = \langle\Gamma|\psi\rangle$ ⇒ Therefore $\mathbf{X}|\Gamma\rangle \neq \langle\Gamma|\mathbf{X}$

Operators

Operators **map, or transform**, a vector in Hilbert space to another vector in that same space

In Quantum Mechanics an ‘operator’ represent measurable

A special class of mapping, generated by operator **X**, have the following property. For some state vector $|\Phi\rangle$

$$\mathbf{X}|\Phi\rangle = \phi|\Phi\rangle$$

Where ϕ is scaler

An equation of this type is called ***eigenvalue*** equation. The vector $|\Phi\rangle$ is called an ***eigenvector*** and the constant ϕ is called ***eigenvalue*** associated with that eigenvector

Matrix Representation

State vector in 2 D vector space

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Spans $\mathbb{C}^2$, the vector space in which qubit lives

States of a qubit in matrix form

$$\boxed{|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad and \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\boxed{\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad and \qquad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}}$$

Therefore sate in matrix form

$$\boxed{|\Psi\rangle = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

Now consider state in 2-D

$$|\varphi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad and \quad |\psi\rangle = \begin{pmatrix} c \\ d \end{pmatrix} \xrightarrow{\text{Inner product}} \langle\varphi|\psi\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d$$

State in dimension

$$|\varphi\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad and \quad |\psi\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \xrightarrow{\text{Inner product}} \langle\varphi|\psi\rangle = \begin{pmatrix} a_1^* & \dots & a_n^* \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1^*b_1 \quad \dots \quad a_n^*b_n$$

$$\langle\varphi|\psi\rangle = \sum_{i=1}^{n} a_1^*b_i$$

Matrix Representation of Operators

State vector in 2 D vector space

$$|\psi\rangle = c|0\rangle + \mathrm{d}|1\rangle$$

We define $\boxed{|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad and \quad |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}}$

$$\boxed{|\Psi\rangle = c\begin{pmatrix}1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}}$$

$$\boxed{\langle 0| = (1 \quad 0) \qquad and \qquad \langle 1| = (0 \quad 1)}$$

$$|\varphi\rangle = a|0\rangle + \mathrm{b}|1\rangle$$

Outer product in matrix form $\quad |\varphi\rangle = \begin{pmatrix}a\\b\end{pmatrix} \quad and \quad |\psi\rangle = \begin{pmatrix}c\\d\end{pmatrix}$

$$|\varphi\rangle\langle\psi| = \begin{pmatrix}a\\b\end{pmatrix}(c^* \quad d^*) = \begin{pmatrix}ac^* & ad^*\\ bc^* & bd^*\end{pmatrix}$$

Matrix representation of an operator

Let us check! $\quad \boldsymbol{X}|0\rangle = \begin{pmatrix}ac^* & ad^*\\ bc^* & bd^*\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}ac^*\\ bc^*\end{pmatrix}$

OR

$$|\varphi\rangle\langle\psi|0\rangle = c^*|\varphi\rangle \quad c^*|\varphi\rangle = c^*\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}ac^*\\ bc^*\end{pmatrix}$$

Adjoint, Hermitian and Unitary Operators

For **operator** $\mathbf{X}$ and ket $|\Phi\rangle$, the **dual of** $\mathbf{X}|\Phi\rangle$ is given by the expression the $\langle\Phi|\boldsymbol{X}^{\dagger}$, $\boldsymbol{X}^{\dagger}$ is called the ***adjoint* or conjugate transpose**, operator to $\mathbf{X}$

Hermitian operator $\mathbf{X}$ that have the property $\mathbf{X} = \boldsymbol{X}^{\dagger}$

X is **Unitary operator** if $\boldsymbol{X}^{\dagger} = \boldsymbol{X}^{-1}$

From matrix algebra we know $\mathbf{X}\boldsymbol{X}^{-1} = \boldsymbol{X}^{-1}\,\mathbf{X} = \boldsymbol{I}$ $\Rightarrow$ $\mathbf{X}\boldsymbol{X}^{\dagger} = \boldsymbol{X}^{\dagger}\,\mathbf{X} = \boldsymbol{I}$ **Identity/unitary operator**

However **usually we represent unitary operators by U** $\Rightarrow$ $\mathbf{U}\boldsymbol{U}^{\dagger} = \boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I}$

Where '$\mathbf{I}$' is **unit operator** $\mathbf{I}|\Phi\rangle = |\Phi\rangle$ For all $|\Phi\rangle$ in Hilbert space

Both **Hermitian and Unitary operator play central role** in **Quantum Computing and Information (QIC) applications**

Adjoint of operators and ket-bra

$$(\alpha\hat{A})^\dagger = \alpha^* A^\dagger \tag{3.22}$$

$$(|\psi\rangle)^\dagger = \langle\psi| \tag{3.23}$$

$$(\langle\psi|)^\dagger = |\psi\rangle \tag{3.24}$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{3.25}$$

$$(\hat{A}|\psi\rangle)^\dagger = \langle\psi|\hat{A}^\dagger \tag{3.26}$$

$$(\hat{A}\hat{B}|\psi\rangle)^\dagger = \langle\psi|\hat{B}^\dagger A^\dagger \tag{3.27}$$

$$(\hat{A}+\hat{B}+\hat{C})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger + \hat{C}^\dagger$$

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Adjoint of **A** $= \boldsymbol{A}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

Definition: Hermitian Operator

An operator $\hat{A}$ is said to be *Hermitian* if

$$\hat{A} = \hat{A}^\dagger$$

$$A = A^\dagger = \begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix}$$

Where a, b, c, and d are real numbers

Example 3.4

Find the adjoint of the operator $\hat{A} = 2|0\rangle\langle 1| - i|1\rangle\langle 0|$.

Solution

First we note that (3.29) tells us that

$$\hat{A}^{\dagger} = (2|0\rangle\langle 1|)^{\dagger} - (i|1\rangle\langle 0|)^{\dagger}$$

We can compute the adjoint of each term by taking the complex conjugate of the constants in each expression and then applying (3.28). We find that

$$\hat{A}^{\dagger} = 2|1\rangle\langle 0| + i|0\rangle\langle 1|$$

Definition: Hermitian Operator

An operator $\hat{A}$ is said to be *Hermitian* if

$$\hat{A} = \hat{A}^{\dagger}$$

Definition: Unitary Operator

The inverse of an operator A is denoted by A^{-1}. This operator satisfies $AA^{-1} = A^{-1}A = I$, where I is the identity operator. An operator is said to be unitary if its adjoint is equal to its inverse. Unitary operators are often denoted using the symbol U and we can state its definition as

$$UU^{\dagger} = U^{\dagger}U = I \tag{3.34}$$

Unitary operators are important because they describe the time evolution of a quantum state. The Pauli operators are both Hermitian *and* Unitary.

Definition: Normal Operator

An operator A is said to be *normal* if

$$AA^{\dagger} = A^{\dagger}A \tag{3.35}$$

Later in the chapter when we consider the commutator of two operators, we will see that this means a normal operator is one that commutes with its adjoint. Hermitian and unitary operators are normal.

Hermitian Operators (Theorems)

Theorem 1.1 *The* ***eigenvalues*** *of a* ***Hermitian operator*** *are* ***real numbers****.*

Theorem 1.2 *If the* ***eigenvalues of Hermitian operator*** *are* **distinct***, then the corresponding eigenvectors are mutually orthogonal. If some of the eigenvalues are not distinct, or* degenerate*, then a* ***linear combination of that subset of eigenvectors can be made to be mutually orthogonal****.*

$$\mathbf{H}|\Phi_n\rangle = \phi_n|\Phi_n\rangle$$

If every value of ϕ_n is different then corresponding $|\Phi_n\rangle$ must be mutually orthogonal

$$\mathbf{H}|\Phi_1\rangle = \phi_1|\Phi_1\rangle$$

$$\mathbf{H}|\Phi_2\rangle = \phi_1|\Phi_2\rangle$$

$|\Phi_1\rangle$ and $|\Phi_2\rangle$ are not orthogonal

Postulates of Quantum Mechanics

Postulate I Kets $|0\rangle, |1\rangle$ constitute a basis for the qubit Hilbert space. An n-qubit register is spanned by basis vectors that are direct products of n-qubits $|a\rangle \otimes |b\rangle \otimes |c\rangle \ldots |n\rangle$, where $a, b, c \cdots n \in 0, 1$.

Postulate II A full description of the system is encapsulated by a vector $|\Psi\rangle$, of unit length, in this 2^n dimensional Hilbert space.

Postulate III (Born's rule) The act of measurement associated with Hermitian operator $\mathbf{A}$ results in one of its eigenvalues. The probability for obtaining a nondegenerate eigenvalue a is given by the expression $|\langle a|\Psi\rangle|^2$ where $|a\rangle$ is an eigenvector of $\mathbf{A}$ that corresponds to eigenvalue a. If the eigenvalue a is degenerate, the probability to find that value is $\sum_i |\langle a_i|\Psi\rangle|^2$ where the sum is over all i in which $a_i = a$.

Postulate IV (collapse hypothesis) Immediately after measurement by $\mathbf{A}$ with result a, the system is described, up to an undetermined phase, by state vector $|a\rangle$. If a is degenerate, the system is in a linear combination of the corresponding eigenvectors.

Operators

Like Differential, Integral, Sum, Difference etc. operators, we have an idea of operators in Q.M

Important postulate of Quantum theory is that there is an operator that corresponds to each physical observable

When operators act on a state of a quit it may change it. e.g.

$$\hat{A}|\psi\rangle = |\phi\rangle \qquad \langle\mu|\hat{A} = \langle\nu|$$

Operators are linear $\quad \hat{A}[\alpha|\psi_1\rangle + \beta|\psi_2\rangle] = \alpha\hat{A}|\psi_1\rangle + \beta\hat{A}|\psi_2\rangle$

$\hat{I}$ is identity operator $\quad \hat{I}|\psi\rangle = |\psi\rangle$

Eigen Operator and Eigen States

Eigen operator

Eigen value

$$\hat{A}|\psi\rangle = a|\psi\rangle$$

Eigen vector

$$\hat{A}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

Pauli Matrices

Set of operators that plays a important role in Quantum Computation

In Q.M dynamical variables like position, momentum , angular momentum, energy etc. are called 'observables'

In Q.M, for each observable there are operator corresponds to each physical variable

Four Pauli Operators

$\sigma_o = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ $\sigma_o\,|0\rangle = |0\rangle$; $\sigma_o\,|1\rangle = |1\rangle$ Identity operator

$\sigma_1 = \sigma_x = \boldsymbol{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow$ $\sigma_1\,|0\rangle = |1\rangle$; $\sigma_1\,|1\rangle = |0\rangle$

$\sigma_2 = \sigma_y = \boldsymbol{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\Rightarrow$ $\sigma_2\,|0\rangle = -i|0\rangle$; $\sigma_2\,|1\rangle = i|0\rangle$

$\sigma_3 = \sigma_z = \boldsymbol{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\Rightarrow$ $\sigma_3\,|0\rangle = |0\rangle$; $\sigma_3\,|1\rangle = -|1\rangle$

Which of the σ-operators are 'Eigen operator'

Which of the σ-operators are 'Hermitian'

All Pauli matrices play important role in **generating operators** in the **Hilbert space of a Qubit**

Most general Self-adjoint (Hermitian) Matric

2 x 2 general Hermitian matrix $\begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix}$ Where a, b, c, d are real numbers

We can re-write above said self-adjoint/Hermitian matrix

$$\begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix} = b\boldsymbol{\sigma}_x + c\boldsymbol{\sigma}_y + \alpha\boldsymbol{\sigma}_z + \beta\mathbb{I}$$

Where $\alpha = \frac{a-d}{2}, \beta = \frac{a+d}{2}, and\ \mathbb{I}$ is the 2 x 2 identity matrix.

Therefore, arbitrary 2 x 2 Hermitian matrix can be represented by a linear combination of three (/four) Pauli matrices and the identity matrix.

The four (Pauli) matrices form a basis for the linear vector space of all 2 x 2 Hermitian matrices.

The Pauli matrices serve as generators of all 2 x 2 unitary matrices

$$\boxed{\begin{aligned} \sigma_X \sigma_Y - \sigma_Y \sigma_X &= 2i\sigma_Z \\ \sigma_Y \sigma_Z - \sigma_Z \sigma_Y &= 2i\sigma_X \\ \sigma_Z \sigma_X - \sigma_X \sigma_Z &= 2i\sigma_Y \end{aligned}} \Rightarrow \boxed{[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k}$$

Verify above using $[A, B] = AB - BA$

Operators in Matrix Form

$$\hat{A} = \begin{pmatrix} \langle u_1|\hat{A}|u_1\rangle & \langle u_1|\hat{A}|u_2\rangle & \cdots & \langle u_1|\hat{A}|u_n\rangle \\ \langle u_2|\hat{A}|u_1\rangle & \langle u_2|\hat{A}|u_2\rangle & & \vdots \\ & & \ddots & \\ \langle u_n|\hat{A}|u_1\rangle & \cdots & & \langle u_n|\hat{A}|u_n\rangle \end{pmatrix} \tag{3.17}$$

Matrix form of operator
is base dependent

$$\hat{A} = \begin{pmatrix} \langle v_1|\hat{A}|v_1\rangle & \langle v_1|\hat{A}|v_2\rangle & \cdots & \langle v_1|\hat{A}|v_n\rangle \\ \langle v_2|\hat{A}|v_1\rangle & \langle v_2|\hat{A}|v_2\rangle & & \vdots \\ & & \ddots & \\ \langle v_n|\hat{A}|v_1\rangle & \cdots & & \langle v_n|\hat{A}|v_n\rangle \end{pmatrix}$$

For single qubit operator in matrix

$$\hat{A} = \begin{pmatrix} \langle 0|\hat{A}|0\rangle & \langle 0|\hat{A}|1\rangle \\ \langle 1|\hat{A}|0\rangle & \langle 1|\hat{A}|1\rangle \end{pmatrix}$$

For two qubit operator in matrix

$$\hat{A} = \begin{pmatrix} \langle 00|\hat{A}|00\rangle & \langle 00|\hat{A}|01\rangle & & \langle 00|\hat{A}|11\rangle \\ \langle 01|\hat{A}|00\rangle & & & \\ & & & \\ \langle 11|\hat{A}|00\rangle & \langle 11|\hat{A}|01\rangle & & \langle 11|\hat{A}|11\rangle \end{pmatrix}$$

Similar matrix for three/four qubit gate

Problem -08

Express following Pauli operators in matrix form

$\sigma_x\,|0\rangle = |1\rangle;\qquad \sigma_x\,|1\rangle = |0\rangle$

$\sigma_y\,|0\rangle = -i|0\rangle;\qquad \sigma_y\,|1\rangle = i|0\rangle$

$\sigma_z\,|0\rangle = |0\rangle;\qquad \sigma_3\,|1\rangle = -|1\rangle$

Express Hadamard H gate in matrix form

$\mathrm{H}|0\rangle = |+\rangle$
$\mathrm{H}|1\rangle = |-\rangle$

Express CNOT operator in matrix form, where

$\mathrm{CNOT}|00\rangle = |00\rangle$
$\mathrm{CNOT}\,|01\rangle = |01\rangle$

$\mathrm{CNOT}|10\rangle = |11\rangle$
$\mathrm{CNOT}\,|11\rangle = |10\rangle$

Eigen Values and eigen state

$$\hat{A}|\psi\rangle = a|\psi\rangle$$

For a given operator **A** how to find eigen values and Eigen state

- Solve the characteristic equation to find the eigenvalues.

$$\hat{A} - \lambda\hat{I} \quad \Rightarrow \quad \det\left|\hat{A} - \lambda\hat{I}\right| = 0$$

- For each eigenvalue, use the eigenvalue equation to generate relations among the components of the given eigenvector.

- Use the normalization condition to find the values of those components.

Problem -09

Find the Eigen values and Eigen states for operator **A**

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\hat{A} - \lambda\hat{I} = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & -\lambda \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda[-\lambda(1-\lambda) - 0] - 0 + i[(0 - (-i)(1-\lambda)] = 0$$

$$\Rightarrow \quad \lambda^2(1-\lambda) - (1-\lambda) = 0 \quad \Rightarrow \quad (1-\lambda)(\lambda^2 - 1) = 0$$

Therefore eigen values are $\lambda = -1, 1, 1$

Now for **each eigen value** we will find eigen vector from characteristic equation

Let $|\varphi_1\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\hat{A}|\varphi_1\rangle = \varphi_1|\varphi_1\rangle \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (-1)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} ic \\ b \\ -ia \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} ic &= -a \\ b &= -b \\ -ia &= -c \end{aligned} \quad \Rightarrow \quad b = 0 \quad \Rightarrow \quad |\varphi_1\rangle = \begin{pmatrix} a \\ 0 \\ ia \end{pmatrix}$$

$|\varphi_1\rangle$ must be normalized $\quad (a)(a) + 0 + (ia)(-ia) = 1 \quad \Rightarrow \quad a^2 + a^2 = 1 \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}}$

Therefore $\quad |\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \quad$ Eigen state for eigen value of '-1'

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

Let $|\varphi_2\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\hat{A}|\varphi_2\rangle = \varphi_2|\varphi_2\rangle$$

$$\begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (1)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} ic \\ b \\ -ia \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$ic = a$$
$$b = b$$
$$-ia = c$$

$$|\varphi_1\rangle = \begin{pmatrix} a \\ b \\ -ia \end{pmatrix}$$

$|\varphi_1\rangle$ must be normalized

$$(a)(a) + (b)(b) + (-ia)(ia) = 1$$

$$2a^2 + b^2 = 1$$

if

$\mathrm{b} = 0 \qquad a = \frac{1}{\sqrt{2}}$

$a = 0 \qquad \mathrm{b} = 1$

Eigen state for eigen value of ‘+1’

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} \qquad |\varphi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

SPECTRAL DECOMPOSITION

An operator A belonging to some vector space that is normal and has a diagonal matrix representation with respect to some basis of that vector space. This result is known as the spectral decomposition theorem. Suppose that an operator A satisfies the spectral decomposition theorem for some basis $|u_i\rangle$. This means that we can write the operator in the form

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i| \tag{3.38}$$

Simple if you know the eigen values of an operator, then matrix form of the operator is simply a diagonal matrix – diagonal elements are simply the eigen values

Example 3.5

Find the eigenvalues of an operator with matrix representation

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution

First we construct the matrix $A - \lambda I$:

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} \end{aligned}$$

Then we compute the determinant

$$\begin{aligned} \det|A - \lambda I| &= \det\begin{vmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - (-1)(1) \\ &= -2 + \lambda - 2\lambda + \lambda^2 + 1 \end{aligned}$$

Rearranging and combining terms, and setting them equal to zero, we obtain

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

THE TRACE OF AN OPERATOR

If an operator is in a matrix representation, the *trace* of the operator is the sum of the diagonal elements. For example,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Tr(A) = a + d$$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad Tr(B) = a + e + i$$

If an operator is written down as an outer product, we take the trace by summing over inner products with the basis vectors. If we label a basis $|u_i\rangle$, then

$$Tr(A) = \sum_{i=1}^{n} \langle u_i|A|u_i\rangle$$

Example 3.8

An operator expressed in the $\{|0\rangle, |1\rangle\}$ basis is given by

$$A = 2i\,|0\rangle\langle 0| + 3|0\rangle\langle 1| - 2|1\rangle\langle 0| + 4|1\rangle\langle 1|$$

Find the trace.

$$Tr(A) = 2i + 4$$

$$Tr(A) = \sum_{i=1}^{n} \langle u_i|A|u_i\rangle$$

Important Properties of the Trace

The trace has some important properties that are good to know. These include the following:

- The trace is *cyclic*, meaning that $Tr(ABC) = Tr(CAB) = Tr(BCA)$.
- The trace of an outer product is the inner product $Tr(|\phi\rangle\langle\psi|) = \langle\phi|\phi\rangle$.
- By extension of the above it follows that $Tr(A|\psi\rangle\langle\phi|) = \langle\phi|A|\psi\rangle$.
- The trace is *basis independent*. Let $|u_i\rangle$ and $|v_i\rangle$ be two bases for some Hilbert space. Then $Tr(A) = \sum\langle u_i|A|u_i\rangle = \sum\langle v_i|A|v_i\rangle$.
- The trace of an operator is equal to the sum of its eigenvalues. If the eigenvalues of A are labeled by λ_i, then $Tr(A) = \sum_{i=1}^{n} \lambda_i$.
- The trace is linear, meaning that $Tr(\alpha A) = \alpha Tr(A)$, $Tr(A+B) = Tr(A) + Tr(B)$.

THE EXPECTATION VALUE OF AN OPERATOR

The *expectation value* of an operator is the *mean* or *average value* of that operator with respect to a given quantum state. In other words, we are asking the following question: If a quantum state $|\psi\rangle$ is prepared many times, and we measure a given operator A each time, what is the average of the measurement results?

This is the expectation value and we write this as

Expectation value of A

$$\langle A \rangle = \langle \psi | A | \psi \rangle \tag{3.39}$$

Expectation value of A^2

$$\langle A^2 \rangle = \langle \psi | A^2 | \psi \rangle$$

Uncertainty in measurement of A

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

Example 3.12

A quantum system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

What is the average or expectation value of X in this state?

The Bloch Sphere

A vector representing the state of a quantum system could look something like arrow, enclosed inside the Bloch sphere, which is the so-called **"state space"** of all possible points to which our state vectors can "point"

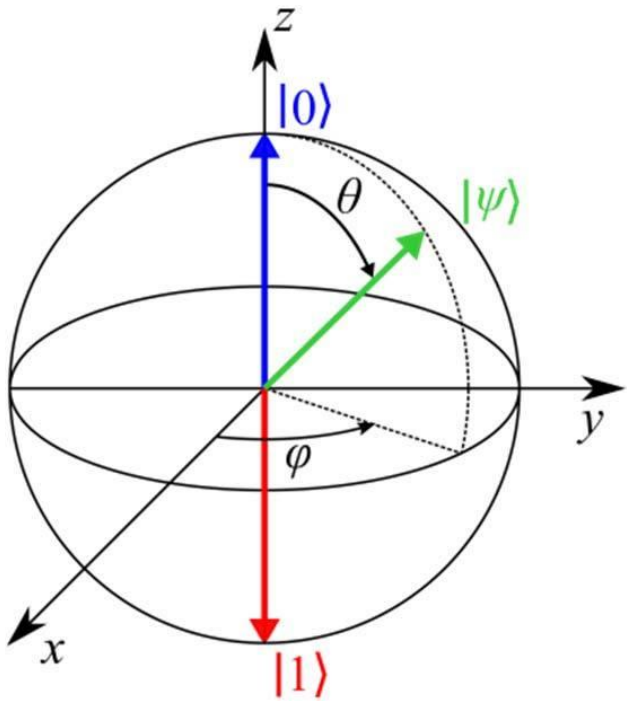
Our state vectors are allowed to **rotate anywhere** on the surface of the sphere, and each of these points represents a different quantum state.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \binom{\alpha}{\beta}$$

α and β are complex numbers

$$\langle\psi|\psi\rangle = (\alpha^* \quad \beta^*)\binom{\alpha}{\beta} = \alpha^2 + \beta^2$$

$\langle\psi|\psi\rangle = \alpha^2 + \beta^2 = 1$ when state is normalized



α and β are complex numbers, therefore

$\alpha = x_0 + ix_1$ and $\beta = x_2 + ix_3$ Where all x's are real numbers

If $|\psi\rangle$ is a normalized sate, then $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$

This equation describe a 3-sphere embedded in a 4-D space

x, y, z are just for reference, actually **Qubit** lives in Hilbert space NOT in 2D or 3D Euclidean space

The Bloch Sphere

A vector representing the state of a quantum system could look something like arrow, enclosed inside the Bloch sphere, which is the so-called **"state space"** of all possible points to which our state vectors can "point"

If $|\psi\rangle$ is a normalized sate, then $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$

This equation describe a 3-sphere embedded in a 4-D space with center at origin

of a point x_0, x_1, x_2, x_3 in this space to a point (x, y, z) in a three-dimensional space where

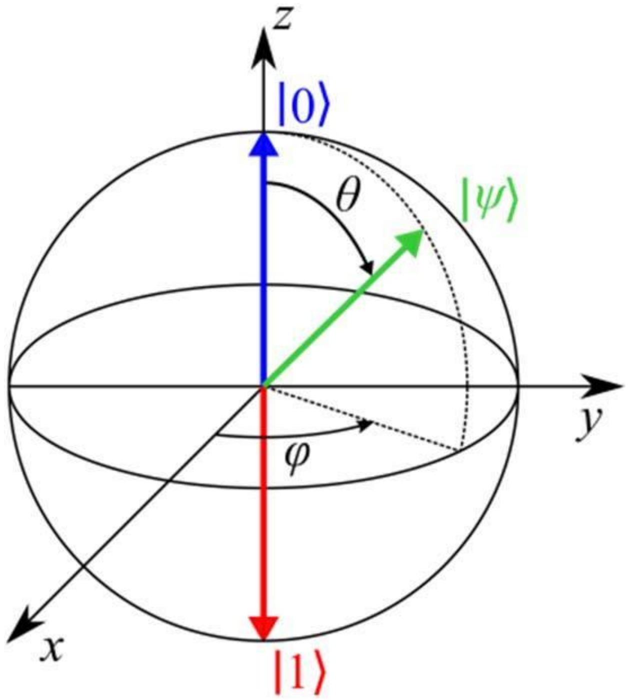
$$
\begin{aligned}
x &= 2(x_0 x_2 + x_1 x_3) \\
y &= 2(x_3 x_0 - x_1 x_2) \\
z &= x_0^2 + x_1^2 - x_2^2 - x_3^2.
\end{aligned}
\tag{2.16}
$$

$$\boxed{\sqrt{x^2 + y^2 + z^2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = 1,}$$

$$
\begin{aligned}
x_0 &= \cos(\theta/2)\cos(\beta) \\
x_1 &= \cos(\theta/2)\sin(\beta) \\
x_2 &= \sin(\theta/2)\cos(\beta + \phi) \\
x_3 &= \sin(\theta/2)\sin(\beta + \phi)
\end{aligned}
$$

for $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \beta \leq 2\pi$ we find that

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$



Standard parameterization of a unit 2-sphere in spherical coordinate system. Here θ and ϕ are polar and azimuthal angles, respectively

The Bloch Sphere

A vector representing the state of a quantum system could look something like arrow, enclosed inside the Bloch sphere, which is the so-called **"state space"** of all possible points to which our state vectors can "point"

Normalized state $|\psi\rangle$ on Bloch sphere can be written as

$$|\psi\rangle = e^{i\beta}\begin{pmatrix} cos\theta/2 \\ e^{i\phi}sin\theta/2 \end{pmatrix} = e^{i\beta}\left[cos\theta/2|0\rangle + e^{i\phi}sin\theta/2|1\rangle\right]$$

Remember in ket-bra notation it was $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \binom{\alpha}{\beta}$

Let us check the Blue state vector $|\mathbf{0}\rangle$

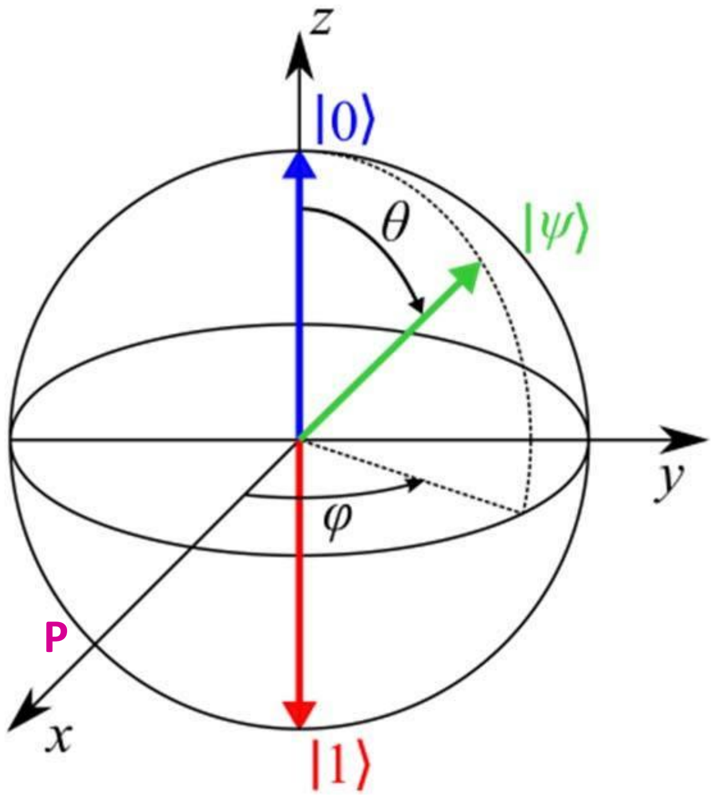
$\theta = 0$, what about ϕ=?

$$|\psi\rangle = \left[cos\theta/2|0\rangle + e^{i\phi}sin\theta/2|1\rangle\right] = cos0|0\rangle + e^{i\phi}sin0|1\rangle = |\mathbf{0}\rangle$$

Let us find the state vector of location **'P'**

$\theta = \pi/2$, what about ϕ=0

$$|\psi\rangle = \left[cos\theta/2|0\rangle + e^{i\phi}sin\theta/2|1\rangle\right] = cos\pi/4|0\rangle + e^{i0}sin\pi/4|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

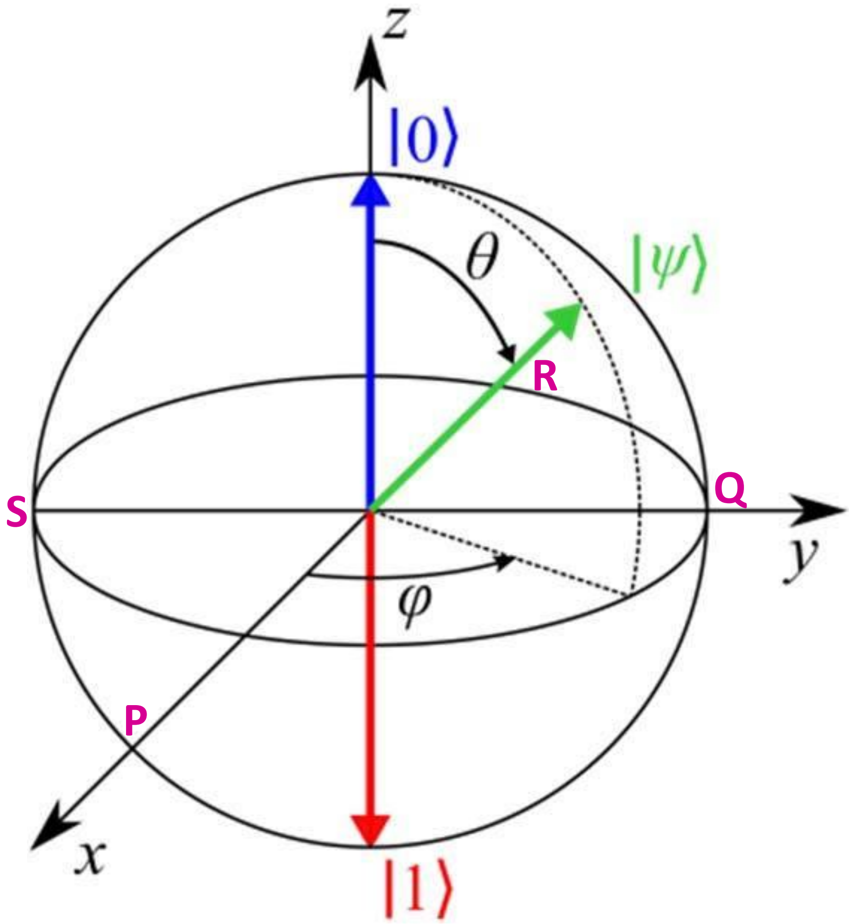


The Bloch Sphere

A vector representing the state of a quantum system could look something like arrow, enclosed inside the Bloch sphere, which is the so-called **"state space"** of all possible points to which our state vectors can "point"

$$|\psi\rangle = e^{i\beta}\begin{pmatrix} cos\theta/2 \\ e^{i\phi} sin\theta/2 \end{pmatrix} = e^{i\beta}\left[cos\theta/2|0\rangle + e^{i\phi} sin\theta/2|1\rangle\right]$$

$e^{i\beta}$ is called overall phase factor and can not be measured. State of a Qubit can be specified by 'a point' on the block sphere, ignoring the 'overall' phase factor



Find the state vector of location **'Q', 'R', 'S'**
(**'R'** is a point where –ve x-axis cuts the Bloch sphere)

The sate vector $|0\rangle$, $|1\rangle$, and at points P, Q, R, S plays very important role in **Quantum Computing and Information** processing

Back up

Let us consider new states (very useful in Quantum Information processing)

$$|u\rangle = |+\rangle \equiv \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |v\rangle = |-\rangle \equiv \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

In matrix form $|u\rangle = |+\rangle \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\qquad |v\rangle = |-\rangle \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$c_1|u\rangle + c_2|v\rangle \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}c_1 + c_2\\c_1 - c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$ Only if $c_1 = c_2 = 0$

Verify linear independence using inner product

Therefore $|u\rangle$ and $|v\rangle$are linearly independent

Now let us find Pauli matrix from basis $|+\rangle$ and $|-\rangle$

$$\sigma_1 = \sigma_x = \boldsymbol{X} \equiv |+\rangle +| - |-\rangle\langle -|$$

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Direct and Kronecker Product

The Direct or Tensor product

Direct product of two kets $|a\rangle$ and $|b\rangle$ is given by $\quad |a\rangle \otimes |b\rangle$

If both kets $|a\rangle$ and $|b\rangle$ are single qubit basis kets $\quad |a\rangle \otimes |b\rangle \equiv \; |0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle$

Above four outer product vectors are orthonormal?

$$(\langle 0| \otimes \langle 0|)(|0\rangle \otimes |0\rangle) = \langle 0|0\rangle\langle 0|0\rangle = 1$$

$$(\langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) = \langle 0|1\rangle\langle 0|0\rangle = 0$$

Yes, above states are mutually orthogonal

We can simplify the notation

$$|a\rangle \otimes |b\rangle \equiv \; |0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle \;\Rightarrow\; |ab\rangle \equiv \; |00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle$$

$$\langle \Psi|\Phi\rangle = (c_1^*\langle b| \otimes \langle a| + c_2^*\langle d| \otimes \langle c|)(d_1|a\rangle \otimes |b\rangle + d_2|c\rangle \otimes |d\rangle) \;\Rightarrow\; \langle \Psi|\Phi\rangle = (c_1^*\langle ba| + c_2^*\langle dc|)(d_1|ab\rangle + d_2|cd\rangle)$$

Make sure you know how to Xply

$$\langle 001|acb\rangle = \langle 0|b\rangle\,\langle 0|c\rangle\langle 1|a\rangle$$

Direct and Kronecker Product

The outer product with multiple qubits

Multi-qubit operators $|ab\rangle = |a\rangle \otimes |b\rangle$ $|cd\rangle = |c\rangle \otimes |d\rangle$

$|ab\rangle\langle cd| = (|a\rangle \otimes |b\rangle)(\langle d| \otimes \langle c|) \equiv |a\rangle\langle c| \tilde{\otimes} |b\rangle\langle d|$ $\tilde{\otimes}$ is called the *Kronecker* product

When this operator on a state $|\Psi\rangle$ $|\Psi\rangle = c_1|01\rangle + c_2|10\rangle$

$$(|ab\rangle\langle cd|)|\Psi\rangle = (|a\rangle \otimes |b\rangle)(\langle d| \otimes \langle c|)(c_1|01\rangle + c_2|10\rangle)$$

$$(|ab\rangle\langle cd|)|\Psi\rangle = \big(|a\rangle\langle c| \tilde{\otimes} |b\rangle\langle d|\big)(c_1|01\rangle + c_2|10\rangle)$$

$$|ab\rangle\langle cd|)|\Psi\rangle = c_1(|a\rangle\langle c| \tilde{\otimes} |b\rangle\langle d|)|01\rangle + c_2(|a\rangle\langle c| \tilde{\otimes} |b\rangle\langle d||10\rangle$$

$$(|ab\rangle\langle cd|)|\Psi\rangle = ((|ab\rangle\langle cd|)(c_1|01\rangle + c_2|10\rangle) = |ab\rangle[c_1\langle cd||01\rangle + c_2\langle cd||10\rangle$$

$$|ab\rangle\langle cd|)|\Psi\rangle = |ab\rangle[c_1\langle c|0\rangle\langle d|1\rangle + c_2\langle c|1\rangle\langle d|0\rangle] = |ab\rangle[c_1\langle cd|10\rangle + c_2\langle cd|10\rangle]$$