# Assignment – 01 - PHYS 514

## Due Monday 19 January – 12:00 mid night

*(Late submission will not be accepted)*

**Submit Question # 3, 4, 6, and 7 ONLY**

Please submit your solution via BB or email. Ensure that your **full name and student ID** are included on your submission

**Q.1 *Plasma Frequency***

Following expression is to find to find the 'plasma frequency - $\omega_p$'

$$\omega_p^2 = \frac{ne^2}{m\epsilon_o}$$

Where 'm' is the effective mass of electron (you use rest mass of electron) and 'n' electron density.

(a) Find the plasma frequency in a metal (of your choice)

(b) Find the temperature corresponding to your calculated value of $\omega_p$.

(Remember $E = hf = h\frac{2\pi}{\omega}$ and $E = K_B T$)

## *Solution*

(a) Electron density for Cu $= 8.5 \times 10^{28}\ m^{-3}$

$$\omega_p^2 = \frac{ne^2}{m\epsilon_o} = \frac{8.5 \times 10^{28}(1.6 \times 10^{-19})^2}{9.1 \times 10^{-31}(8.85 \times 10^{-12})}$$

$$\omega_p =$$

(b) The required temperature T

$$k_B T = \hbar\omega$$

$$T = \frac{1}{k_B}\hbar\omega = \frac{1}{1.38 \times 10^{-23}}\frac{6.63 \times 10^{-34}}{2\pi}\omega_p$$

**Q.2**

Briefly explain the concept of 'skin depth', in reference to electromagnetic radiations

Ask AI!

**Q.3** ***Commutation Relation***
Using the commutation relation between $\hat{x}$ and $\hat{p}$ show that

$$[a, a^\dagger] = 1$$

## Solution

Using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

$$i\hbar = [\hat{x}, \hat{p}]$$

$$i\hbar = i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}}[(\hat{a}^\dagger + \hat{a}), (\hat{a}^\dagger + \hat{a})\ ]$$

$$i\hbar = i\frac{\hbar}{2}\{[\hat{a}^\dagger, \hat{a}^\dagger] - [\hat{a}^\dagger, \hat{a}] + [\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}\ ]\}$$

$$i\hbar = i\frac{\hbar}{2}\{-[\hat{a}^\dagger, \hat{a}] + [\hat{a}, \hat{a}^\dagger]\}$$

$$1 = \frac{1}{2}\{[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\}$$

$$1 = [\hat{a}, \hat{a}^\dagger]$$

**Q.4** ***Eigen State of Oscillator Hamiltonian***
If

$$\hat{H}|n\rangle = E_n|n\rangle$$

Where $|n\rangle$ are the Fock states of LC oscillator, using commutation relation between $a$ and $a^\dagger$ show that

$$\hat{H}\hat{a}|n\rangle = (E_n - \hbar\omega)\hat{a}|n\rangle$$

## Solution

Consider LHS

$$\hat{H}\hat{a}|n\rangle = \hbar\omega[\hat{a}^\dagger\hat{a} + 1/2]\hat{a}|n\rangle$$

$$\hat{H}\hat{a}|n\rangle = \hbar\omega[\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}1/2]|n\rangle$$

Using commutation relation

$$1 = [\hat{a}, \hat{a}^{\dagger}] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}a$$

$$\hat{a}^{\dagger}\hat{a} = \hat{a}\hat{a}^{\dagger} - 1$$

**Therefore**

$$\hat{H}\hat{a}|n\rangle = \hbar\omega[(\hat{a}\hat{a}^{\dagger} - 1)\hat{a} + \hat{a}1/2]|n\rangle$$

$$\hat{H}\hat{a}|n\rangle = \hbar\omega\hat{a}\{[\hat{a}^{\dagger}\hat{a} + 1/2] - 1\}|n\rangle$$

$$\hat{H}\hat{a}|n\rangle = \hat{a}(\hbar\omega[\hat{a}^{\dagger}\hat{a} + 1/2]|n\rangle) - \hbar\omega\hat{a}|n\rangle$$

$$\hat{H}\hat{a}|n\rangle = \hat{a}E_n|n\rangle - \hbar\omega\hat{a}|n\rangle$$

$$\hat{H}\hat{a}|n\rangle = (E_n - \hbar\omega)\hat{a}|n\rangle$$

**Q.5 *Hamiltonian and ladder operator***

Using commutation relation of the ladder operator, show that

$$\hat{H} = \frac{\hbar\Omega}{2}[\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}]$$

can be expressed as $\hat{H} = \hbar\Omega[\hat{a}^{\dagger}\hat{a} + 1/2]$

## Solution

Using commutation relation

$$1 = [\hat{a}, \hat{a}^{\dagger}] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}$$

$$\hat{a}\hat{a}^{\dagger} = \hat{a}^{\dagger}\hat{a} + 1$$

Therefore

$$\hat{H} = \frac{\hbar\Omega}{2}[\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}] \quad \Rightarrow \quad \hat{H} = \frac{\hbar\Omega}{2}[\hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{a} + 1]$$

$$\hat{H} = \frac{\hbar\Omega}{2}[2\hat{a}^{\dagger}\hat{a} + 1]$$

$$\hat{H} = \hbar\Omega[\hat{a}^{\dagger}\hat{a} + 1/2]$$

**Q.6 Ladder Operator**

(a) Show that

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$$

$$[\hat{n}, \hat{a}\ ] = -\hat{a}$$

## Solution

Consider LHS

$$[\hat{n}, \hat{a}^\dagger] = \hat{n}\hat{a}^\dagger - \hat{a}^\dagger\hat{n}$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a}$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger([\hat{a}, \hat{a}^\dagger])$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$$

Similarly

$$[\hat{n}, \hat{a}\ ] = \hat{n}\hat{a} - \hat{a}\hat{n}$$

$$[\hat{n}, \hat{a}\ ] = \hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}$$

$$[\hat{n}, \hat{a}\ ] = (\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}$$

$$[\hat{n}, \hat{a}\ ] = (-[\hat{a}, \hat{a}^\dagger])\hat{a}$$

$$[\hat{n}, \hat{a}\ ] = -\hat{a}$$

(b) Find the normalization constant $C_1$ and $C_2$

$$\hat{a}|n\rangle = C_1|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = C_2|n+1\rangle$$

## Solution

Consider

$$\hat{a}|n\rangle = C_1|n-1\rangle$$

Using normalization condition

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n-1|C_1^*C_1|n-1\rangle$$

$$\langle n|\hat{n}|n\rangle = [C_1]^2\langle n-1|n-1\rangle$$

$$\langle n|n|n\rangle = [C_1]^2$$

$$n = [C_1]^2$$

$$C_1 = \sqrt{n}$$

Now Consider

$$\hat{a}^\dagger|n\rangle = C_2|n+1\rangle$$

Using normalization condition

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n+1|C_2^* C_2|n+1\rangle$$

$$\langle n|(\hat{a}^\dagger\hat{a}+1)|n\rangle = [C_2]^2\langle n+1|n+1\rangle$$

$$\langle n|(\hat{n}+1)|n\rangle = [C_2]^2$$

$$\langle n|(n+1)|n\rangle = [C_2]^2$$

$$(n+1)\langle n|n\rangle = [C_2]^2$$

$$[C_2]^2 = n+1$$

$$C_2 = \sqrt{n+1}$$

**Q.7 Number Operator**

Consider a cavity that contains superposition of two Fock states described by

$$|\psi\rangle = \sqrt{0.99}|0\rangle + \sqrt{0.01}|100\rangle$$

a) What is the average number of photons in the cavity?
b) If you annihilate a photon by acting annihilation operator $\hat{a}$ on this state, then how many photons remain in the cavity? Interpret the result.

## Solution

$$\langle n\rangle = \langle\psi|\hat{n}|\psi\rangle$$

$$\langle n\rangle = \left(\langle 0|\sqrt{099} + \langle 100|\sqrt{0.01}\right)|\hat{n}|\left(\sqrt{0.99}|0\rangle + \sqrt{0.01}|100\rangle\right)$$

$$\langle n\rangle = \left(\langle 0|\sqrt{099} + \langle 100|\sqrt{0.01}\right)\left(0 + 100\sqrt{0.01}|100\rangle\right)$$

$$\langle n\rangle = (0.01)100$$

$$\langle n\rangle = 1$$

(b)

Using following property of annihilation operator $\hat{a}$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad and \quad \hat{a}|0\rangle = 0$$

$$\hat{a}|\psi\rangle = \sqrt{0.99}\hat{a}|0\rangle + \sqrt{0.01}\hat{a}|100\rangle$$

$$|\psi'\rangle = \hat{a}|\psi\rangle = 0 + \sqrt{0.01}\sqrt{100}|99\rangle$$

$$|\psi'\rangle = |99\rangle$$

Now number of photons in new state $|\psi'\rangle$

$$\langle n\rangle = \langle\psi'|\hat{n}|\psi'\rangle$$

$$\langle n\rangle = \langle 99|\hat{n}|99\rangle$$

$$\langle n\rangle = 99$$

How come lowering (annihilation) operator increases the average number of photons from 1 to 100! How come? interesting