Q.1 Plasma Frequency

Following expression is to find the 'plasma frequency - ω_p':

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

Where 'm' is the effective mass of electron (use rest mass) and 'n' electron density.

(a) Find the plasma frequency in a metal (of your choice)

(b) Find the temperature corresponding to your calculated value of ω_p.
(Remember $E = hf = \frac{h}{2\pi}\omega$ and $E = k_B T$)

Solution:
(a) Plasma frequency for Copper:
Using $n = 8.5 \times 10^{28}$ m^{-3}, $e = 1.6 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m:

$$\begin{aligned}\omega_p^2 &= \frac{(8.5 \times 10^{28})(1.6 \times 10^{-19})^2}{(9.11 \times 10^{-31})(8.85 \times 10^{-12})} \\ &= \frac{2.176 \times 10^{-9}}{8.06 \times 10^{-42}} = 2.70 \times 10^{32} \text{ rad}^2/\text{s}^2\end{aligned}$$

$$\boxed{\omega_p = 1.64 \times 10^{16} \text{ rad/s}} \quad \Rightarrow \quad f_p = \frac{\omega_p}{2\pi} = \boxed{2.61 \times 10^{15} \text{ Hz}}$$

(b) Corresponding temperature:
Using $E = \hbar\omega = k_B T$:

$$T = \frac{\hbar\omega_p}{k_B} = \frac{(1.055 \times 10^{-34})(1.64 \times 10^{16})}{1.38 \times 10^{-23}} = \boxed{1.25 \times 10^5 \text{ K}}$$

Q.2 Skin Depth

Briefly explain the concept of 'skin depth', in reference to electromagnetic radiations.

Solution:
Skin depth δ is the distance into a conductor at which the amplitude of an electromagnetic wave decreases to $1/e$ of its surface value. For a good conductor:

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

At high frequencies, EM waves penetrate only a thin surface layer. Current flows primarily within this layer, effectively confined to a "skin" on the conductor surface. Higher frequencies and higher conductivity result in smaller skin depth.

Q.3 Commutation Relation

Using the commutation relation between $\hat{x}$ and $\hat{p}$ show that

$$[a, a^\dagger] = 1$$

Solution:
Ladder operators: $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \left[\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}, \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\right] \\
&= \frac{m\omega}{2\hbar}[\hat{x}, \hat{x}] - \frac{i}{2\hbar}[\hat{x}, \hat{p}] + \frac{i}{2\hbar}[\hat{p}, \hat{x}] + \frac{1}{2m\omega\hbar}[\hat{p}, \hat{p}] \\
&= 0 - \frac{i}{2\hbar}(i\hbar) + \frac{i}{2\hbar}(-i\hbar) + 0 \\
&= \frac{1}{2} + \frac{1}{2} = \boxed{1}
\end{aligned}$$

Q.4 Eigenstate of Oscillator Hamiltonian

If

$$\hat{H}\,|n\rangle = E_n\,|n\rangle$$

Where $|n\rangle$ are the Fock states of LC oscillator, using commutation relation between a and $a^\dagger$ show that

$$\hat{H}\hat{a}\,|n\rangle = (E_n - \hbar\omega)\hat{a}\,|n\rangle$$

Solution:
Given: $\hat{H}\,|n\rangle = E_n\,|n\rangle$ with $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$
Using $[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$:

$$\begin{aligned}
[\hat{H}, \hat{a}] &= \hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}] = \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}) \\
&= \hbar\omega(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a} = \hbar\omega(-1)\hat{a} = -\hbar\omega\hat{a}
\end{aligned}$$

Therefore $\hat{H}\hat{a} = \hat{a}\hat{H} - \hbar\omega\hat{a}$

$$\begin{aligned}
\hat{H}\hat{a}\,|n\rangle &= (\hat{a}\hat{H} - \hbar\omega\hat{a})\,|n\rangle \\
&= \hat{a}(E_n\,|n\rangle) - \hbar\omega\hat{a}\,|n\rangle \\
&= \boxed{(E_n - \hbar\omega)\hat{a}\,|n\rangle}
\end{aligned}$$

Q.5 Hamiltonian and Ladder Operator

Using commutation relation of the ladder operator, show that

$$\hat{H} = \frac{\hbar\Omega}{2}[a^\dagger a + aa^\dagger]$$

can be expressed as $\hat{H} = \hbar\Omega[a^\dagger a + 1/2]$

Solution:
Given: $\hat{H} = \frac{\hbar\Omega}{2}[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger]$
Using $[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$:

$$\begin{aligned}
\hat{H} &= \frac{\hbar\Omega}{2}[\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} + 1] \\
&= \frac{\hbar\Omega}{2}[2\hat{a}^\dagger\hat{a} + 1] \\
&= \boxed{\hbar\Omega[\hat{a}^\dagger\hat{a} + 1/2]}
\end{aligned}$$

Q.6 Ladder Operator

(a) Show that

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$$
$$[\hat{n}, \hat{a}] = -\hat{a}$$

(b) Find the normalization constant C_1 and C_2

$$\hat{a}\,|n\rangle = C_1\,|n-1\rangle$$
$$\hat{a}^\dagger\,|n\rangle = C_2\,|n+1\rangle$$

Solution:
(a) With $\hat{n} = \hat{a}^\dagger\hat{a}$:

$$[\hat{n}, \hat{a}^\dagger] = [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger(1) + 0 = \boxed{\hat{a}^\dagger}$$
$$[\hat{n}, \hat{a}] = [\hat{a}^\dagger\hat{a}, \hat{a}] = \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = 0 + (-1)\hat{a} = \boxed{-\hat{a}}$$

(b) For $\hat{a}\,|n\rangle = C_1\,|n-1\rangle$:

$$\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle = |C_1|^2\,\langle n-1|n-1\rangle = |C_1|^2$$
$$\langle n|\,\hat{n}\,|n\rangle = n \quad \Rightarrow \quad \boxed{C_1 = \sqrt{n}}$$

For $\hat{a}^\dagger\,|n\rangle = C_2\,|n+1\rangle$:

$$\langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle = |C_2|^2\,\langle n+1|n+1\rangle = |C_2|^2$$
$$\langle n|\,(\hat{a}^\dagger\hat{a} + 1)\,|n\rangle = n + 1 \quad \Rightarrow \quad \boxed{C_2 = \sqrt{n+1}}$$

Q.7 Number Operator

Consider a cavity that contains superposition of two Fock states described by

$$|\psi\rangle = \sqrt{0.99}\,|0\rangle + \sqrt{0.01}\,|100\rangle$$

(a) What is the average number of photons in the cavity?

(b) If you annihilate a photon by acting annihilation operator $\hat{a}$ on this state, then how many photons remain in the cavity? Interpret the result.

Solution:
(a) Average number of photons:

$$\begin{aligned}\langle\hat{n}\rangle &= \langle\psi|\,\hat{n}\,|\psi\rangle = 0.99\,\langle 0|\,\hat{n}\,|0\rangle + 0.01\,\langle 100|\,\hat{n}\,|100\rangle \\ &= 0.99(0) + 0.01(100) = \boxed{1}\end{aligned}$$

(b) After annihilation:
Using $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}\,|0\rangle = 0$:

$$\hat{a}\,|\psi\rangle = \sqrt{0.99}(0) + \sqrt{0.01}\sqrt{100}\,|99\rangle = |99\rangle$$

The state is unnormalized. Normalizing: $|\psi'\rangle = |99\rangle$
Average photons after: $\langle\hat{n}\rangle' = \boxed{99}$
Interpretation: Annihilating one photon collapses the superposition. The vacuum component $|0\rangle$ vanishes (no photon to remove), leaving only $|99\rangle$. This is a projective measurement effect, the act of removing a photon provides “which-state” information.