Assignment – 02 - PHYS 514

Due Wednesday 28 January – 12:00 mid night

(Late submission will not be accepted)

Submit solutions of problem # 1, 4, and 5 ONLY

Please submit your solution via BB or email. Ensure that your **full name and student ID** are included on your submission

Q.1 [Lagrange and Euler-Lagrange Equation]

(a) Write down the Lagrange equation of LC circuit in terms of q *and* $\dot{q}$.

(b) Using following Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Show that

$$\ddot{q} = -\Omega^2 q$$

Where $\Omega = \frac{1}{\sqrt{LC}}$

(c) Find conjugate momentum for (position) coordinate q (using Lagrange of part a)

(d) Find the Hamiltonian of the LC circuits.

Solution

(a)

$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{1}{2}LI^2 - \frac{1}{2}CV^2$$

$$\mathrm{L} = \frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}$$

(b)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial}{\partial q}\left[\frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}\right] - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}\right]\right) = 0$$

$$Cq + L\ddot{q} = 0$$

$$\ddot{q} = -\frac{1}{LC}q = -\Omega^2 q$$

Q.2 [Hamiltonian of LC Oscillator]

(a) Show that the

$$\hat{H} = \frac{1}{2C}\hat{Q}^2 + \frac{1}{2L}\hat{\varphi}^2$$

can be expressed as

$$\hat{H} = \frac{\hbar\Omega}{2}[a^{\dagger}a + aa^{\dagger}]$$

Where $a = i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi}$ and $a^{\dagger} = -i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\varphi}$

Solution

(a)

$$aa^{\dagger} = \frac{1}{2L\hbar\Omega}\hat{\varphi}^2 + \frac{1}{2C\hbar\Omega}\hat{Q}^2 + \frac{i}{2\hbar\Omega\sqrt{LC}}[\hat{Q}\hat{\varphi} - \hat{\varphi}\hat{Q}]$$

$$aa^{\dagger} = \frac{1}{2L\hbar\Omega}\hat{\varphi}^2 + \frac{1}{2C\hbar\Omega}\hat{Q}^2 + \frac{i}{2\hbar\Omega\sqrt{LC}}(-i\hbar)$$

$$aa^{\dagger} = \frac{1}{\hbar\Omega}\left[\frac{1}{2L}\hat{\varphi}^2 + \frac{1}{2C}\hat{Q}^2\right] + \frac{1}{2}$$

$$aa^{\dagger} - \frac{1}{2} = \frac{1}{\hbar\Omega}[H]$$

$$H = \hbar\Omega\left(aa^{\dagger} - \frac{1}{2}\right)$$

Similarly

$$H = \hbar\Omega\left(a^{\dagger}a + \frac{1}{2}\right)$$

Therefore, by adding last two equations

$$\hat{H} = \frac{\hbar\Omega}{2}[a^{\dagger}a + aa^{\dagger}]$$

Q.3 [Expectation values of Q and ϕ]

Using 'ladder operation' expression for Q, find expectation value for Q $\left(\langle\hat{Q}\rangle\right)$ and Q^2 $\left(\langle\hat{Q}^2\rangle\right)$ for vacuum (zero photon state).

Solution

Expectation value of Q

$$\hat{Q} = -iQ_{ZPF}(\hat{a} - \hat{a}^{\dagger})$$

$$\langle Q\rangle = \langle 0|\hat{Q}|0\rangle$$

$$\langle Q\rangle = \langle 0|[-iQ_{ZPF}(\hat{a} - \hat{a}^{\dagger})]|0\rangle$$

$$\langle Q\rangle = -iQ_{ZPF}\langle 0|(\hat{a} - \hat{a}^{\dagger})|0\rangle$$

$$\langle Q\rangle = -iQ_{ZPF}\{\langle 0|(\hat{a}|0\rangle - \hat{a}^{\dagger}|0\rangle)\}$$

$$\langle Q\rangle = -iQ_{ZPF}\{\langle 0|(0 + |1\rangle)\}$$

$$\langle Q\rangle = 0$$

Expectation value of Q²

$$\langle Q^2\rangle = (-iQ_{ZPF})^2\langle 0|(\hat{a} - \hat{a}^{\dagger})^2|0\rangle$$

$$\langle Q\rangle = -Q_{ZPF}{}^2\{\langle 0|\hat{a}\hat{a} + \hat{a}^{\dagger}\hat{a}^{\dagger} - \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}|0\rangle\}$$

$$\langle Q\rangle = -Q_{ZPF}{}^2\{\langle a|\hat{a}\hat{a}|0\rangle + \langle 0|\hat{a}^{\dagger}\hat{a}^{\dagger}|0\rangle - \langle 0|\hat{a}\hat{a}^{\dagger}|0\rangle - \langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle\}$$

$$\langle Q\rangle = -Q_{ZPF}{}^2\{0 + \langle 0|\sqrt{2}|2\rangle - 1 + 0\}$$

$$\langle Q\rangle = Q_{ZPF}{}^2$$

Q.4 [Zero Point Fluctuation (ZPF)]

Calculate the zero-point fluctuation in charge, Q_{ZPF} for an LC oscillator circuit. Calculate the Φ_{ZPF} as well. Assume L= 50 nH and C = 70 pF. Express Q_{ZPF} in terms of number of electrons (i.e. Q_{ZPF}/e).

Solution

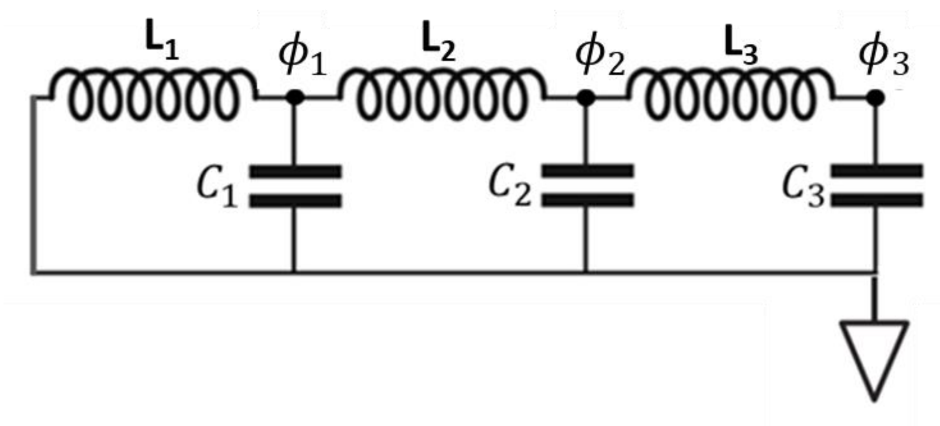
$$Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{50nH}{70pF}} = 26.726\,\Omega$$

$$Q_{ZPF} = \sqrt{\frac{\hbar C\Omega}{2}} = \sqrt{\frac{\hbar}{2Z}} = \sqrt{\frac{6.63\times10^{-34}}{2(2\pi)(26.726)}} = 1.405\times10^{-18}C$$

$$\Phi_{ZPF} = \sqrt{\frac{\hbar L\Omega}{2}} = \sqrt{\frac{\hbar Z}{2}} = \sqrt{\frac{6.63\times10^{-34}(26.726)}{2(2\pi)}} = 3.753\times10^{-17}Wb$$

$$\frac{Q_{ZPF}}{e} \approx 9$$

Q. 5 [Lagrange and Hamiltonian of Quantum Circuit]



(a) Write down the Lagrange of the above circuit in-terms of nodal flux ϕ

(b) Find the momentum conjugate, Q, to the flux ϕ (1, 2, and 3)

(c) Write down the commutation relation between Q and ϕ

(d) Find the Hamiltonian of the above circuit
(Hint: $H = \sum_{1=1}^{3} Q_i\dot{\phi}_i - L$

(e) Write down the Hamiltion's equations of motion

Solution

(a)

$$\mathrm{L} = \frac{1}{2}C_1\dot{\phi}_1^2 + \frac{1}{2}C_2\dot{\phi}_2^2 + \frac{1}{2}C_3\dot{\phi}_3^2 - \left(\frac{1}{2L_1}\phi_1^2 + \frac{1}{2L_2}(\phi_2 - \phi_1)^2 + \frac{1}{2L_2}(\phi_3 - \phi_2)^2\right)$$

(b)

Conjugate momentum

$$Q_i = \frac{\partial L}{\partial \dot{\phi}_i}$$

Therefore

$$Q_1 = \frac{\partial L}{\partial \dot{\phi}_1} = C_1\dot{\phi}_1$$

$$Q_2 = \frac{\partial L}{\partial \dot{\phi}_2} = C_2\dot{\phi}_2$$

$$Q_3 = \frac{\partial L}{\partial \dot{\phi}_3} = C_3\dot{\phi}_3$$

(c)

$$[\phi_i, Q_i] = i\hbar$$

(d)

$$H = \sum_{1=1}^{3} Q_i\dot{\phi}_i - L$$

$$H = Q_1\dot{\phi}_1 + Q_1\dot{\phi}_1 + Q_1\dot{\phi}_1 - \left\{\frac{1}{2}C_1\dot{\phi}_1^2 + \frac{1}{2}C_2\dot{\phi}_2^2 + \frac{1}{2}C_3\dot{\phi}_3^2 + \frac{1}{2L_1}\phi_1^2 + \frac{1}{2L_2}(\phi_2 - \phi_1)^2 + \frac{1}{2L_2}(\phi_3 - \phi_2)^2\right\}$$

$$H = Q_1\frac{Q_1}{C_1} + Q_1\frac{Q_2}{C_2} + Q_1\frac{Q_3}{C_3}$$
$$- \left\{\frac{1}{2}C_1\left(\frac{Q_1}{C_1}\right)^2 + \frac{1}{2}C_2\left(\frac{Q_1}{C_1}\right)^2 + \frac{1}{2}C_3\left(\frac{Q_1}{C_1}\right)^2 + \frac{1}{2L_1}\phi_1^2 + \frac{1}{2L_2}(\phi_2 - \phi_1)^2 + \frac{1}{2L_2}(\phi_3 - \phi_2)^2\right\}$$

$$H = \frac{1}{2C_1}Q_1^2 + \frac{1}{2C_2}Q_2^2 + \frac{1}{2C_3}Q_3^2 - \frac{1}{2L_1}\phi_1^2 - \frac{1}{2L_2}(\phi_2 - \phi_1)^2 - \frac{1}{2L_2}(\phi_3 - \phi_2)^2$$

(a) Hamilton's equations of motion

$$\dot{\phi}_\iota = \frac{\partial H}{\partial Q_i}$$
$$\dot{Q}_\iota = -\frac{\partial H}{\partial \phi_i}$$

Therefore,

$$\dot{\phi}_1 = \frac{\partial H}{\partial Q_1} = \frac{Q_1}{C_1}; \qquad \dot{\phi}_2 = \frac{\partial H}{\partial Q_2} = \frac{Q_2}{C_2}; \qquad \dot{\phi}_1 = \frac{\partial H}{\partial Q_1} = \frac{Q_3}{C_3}$$

$$\dot{Q}_1 = -\frac{\partial H}{\partial \phi_1} = \frac{\phi_1}{L_1};\ \dot{Q}_2 = -\frac{\partial H}{\partial \phi_2} = \frac{(\phi_2 - \phi_1)}{L_2};\ \dot{Q}_3 = -\frac{\partial H}{\partial \phi_3} = \frac{(\phi_3 - \phi_2)}{L_3}$$