Q.1 Lagrange and Euler-Lagrange Equation

(a) Lagrangian:
Inductor stores kinetic energy $T = \frac{1}{2}L\dot{q}^2$, capacitor stores potential energy $U = \frac{q^2}{2C}$:

$$\boxed{L = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}}$$

(b) Equation of motion:
$\frac{\partial L}{\partial q} = -\frac{q}{C}$, $\quad \frac{\partial L}{\partial \dot{q}} = L\dot{q}$, $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = L\ddot{q}$
Euler-Lagrange: $-\frac{q}{C} - L\ddot{q} = 0 \Rightarrow \boxed{\ddot{q} = -\Omega^2 q}$ where $\Omega = \frac{1}{\sqrt{LC}}$
(c) Conjugate momentum:

$$\boxed{p = \frac{\partial L}{\partial \dot{q}} = L\dot{q}}$$

(d) Hamiltonian:
$H = p\dot{q} - L$ with $\dot{q} = p/L$:

$$H = \frac{p^2}{L} - \frac{p^2}{2L} + \frac{q^2}{2C} = \boxed{\frac{p^2}{2L} + \frac{q^2}{2C}}$$

Q.2 Hamiltonian of LC Oscillator

Define $\alpha = \frac{1}{\sqrt{2C\hbar\Omega}}$, $\beta = \frac{1}{\sqrt{2L\hbar\Omega}}$, so $\hat{a} = i\alpha\hat{Q} + \beta\hat{\varphi}$, $\hat{a}^\dagger = -i\alpha\hat{Q} + \beta\hat{\varphi}$
Computing products:

$$\begin{aligned}\hat{a}^\dagger\hat{a} &= \alpha^2\hat{Q}^2 + \beta^2\hat{\varphi}^2 + i\alpha\beta[\hat{Q},\hat{\varphi}]\\ \hat{a}\hat{a}^\dagger &= \alpha^2\hat{Q}^2 + \beta^2\hat{\varphi}^2 - i\alpha\beta[\hat{Q},\hat{\varphi}]\end{aligned}$$

Adding (commutator terms cancel):

$$\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = 2\alpha^2\hat{Q}^2 + 2\beta^2\hat{\varphi}^2 = \frac{\hat{Q}^2}{C\hbar\Omega} + \frac{\hat{\varphi}^2}{L\hbar\Omega}$$

Therefore: $\boxed{\frac{\hbar\Omega}{2}[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger] = \frac{\hat{Q}^2}{2C} + \frac{\hat{\varphi}^2}{2L} = \hat{H}}$

Q.3 Expectation Values of Q and φ

From $\hat{a} - \hat{a}^\dagger = 2i\alpha\hat{Q}$, solve for $\hat{Q}$:

$$\hat{Q} = i\sqrt{\frac{C\hbar\Omega}{2}}(\hat{a}^\dagger - \hat{a})$$

Expectation $\langle\hat{Q}\rangle$: Using $\hat{a}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger = 0$:

$$\langle\hat{Q}\rangle = i\sqrt{\frac{C\hbar\Omega}{2}}\langle 0|(\hat{a}^\dagger - \hat{a})|0\rangle = \boxed{0}$$

Expectation $\langle\hat{Q}^2\rangle$:

$$\hat{Q}^2 = -\frac{C\hbar\Omega}{2}(\hat{a}^\dagger - \hat{a})^2$$

For vacuum, only $\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = 1$ survives (using $[\hat{a},\hat{a}^\dagger] = 1$):

$$\langle\hat{Q}^2\rangle = \boxed{\frac{C\hbar\Omega}{2}}$$

Q.4 Zero Point Fluctuation (ZPF)

Given: $L = 50$ nH, $C = 70$ pF

Characteristic impedance:

$$Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{50 \times 10^{-9}}{70 \times 10^{-12}}} = 26.73\,\Omega$$

Zero-point charge fluctuation ($Q_{\text{ZPF}} = \sqrt{\langle \hat{Q}^2 \rangle}$):

$$Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2Z}} = \sqrt{\frac{1.055 \times 10^{-34}}{53.46}} = \boxed{1.40 \times 10^{-18}\text{ C}}$$

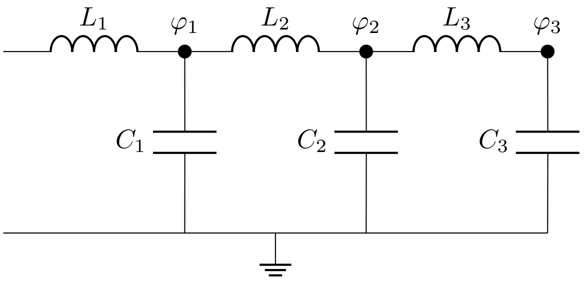
In electron units:

$$\frac{Q_{\text{ZPF}}}{e} = \frac{1.40 \times 10^{-18}}{1.6 \times 10^{-19}} = \boxed{8.8\text{ electrons}}$$

Zero-point flux fluctuation:

$$\Phi_{\text{ZPF}} = \sqrt{\frac{\hbar Z}{2}} = \sqrt{\frac{(1.055 \times 10^{-34})(26.73)}{2}} = \boxed{3.75 \times 10^{-17}\text{ Wb}}$$

Q.5 Lagrange and Hamiltonian of Quantum Circuit



(a) Lagrangian:
Capacitors store kinetic energy $\frac{1}{2}C_i\dot{\varphi}_i^2$, inductors store potential energy $\frac{(\Delta\varphi)^2}{2L}$:

$$\boxed{L = \sum_{i=1}^{3} \frac{C_i\dot{\varphi}_i^2}{2} - \frac{\varphi_1^2}{2L_1} - \frac{(\varphi_2-\varphi_1)^2}{2L_2} - \frac{(\varphi_3-\varphi_2)^2}{2L_3}}$$

(b) Conjugate momenta:

$$Q_i = \frac{\partial L}{\partial \dot{\varphi}_i} \Rightarrow \boxed{Q_1 = C_1\dot{\varphi}_1, \quad Q_2 = C_2\dot{\varphi}_2, \quad Q_3 = C_3\dot{\varphi}_3}$$

(c) Commutation relations:

$$\boxed{[\hat{\varphi}_i, \hat{Q}_j] = i\hbar\delta_{ij}}$$

(d) Hamiltonian:
Using $H = \sum_i Q_i\dot{\varphi}_i - L$ with $\dot{\varphi}_i = Q_i/C_i$:

$$\boxed{H = \sum_{i=1}^{3} \frac{Q_i^2}{2C_i} + \frac{\varphi_1^2}{2L_1} + \frac{(\varphi_2-\varphi_1)^2}{2L_2} + \frac{(\varphi_3-\varphi_2)^2}{2L_3}}$$

(e) Hamilton's equations:
$\dot{\varphi}_i = \frac{\partial H}{\partial Q_i}$: $\boxed{\dot{\varphi}_1 = \frac{Q_1}{C_1}, \quad \dot{\varphi}_2 = \frac{Q_2}{C_2}, \quad \dot{\varphi}_3 = \frac{Q_3}{C_3}}$
$\dot{Q}_i = -\frac{\partial H}{\partial \varphi_i}$:

$$\dot{Q}_1 = \boxed{-\frac{\varphi_1}{L_1} + \frac{\varphi_2-\varphi_1}{L_2}}$$

$$\dot{Q}_2 = \boxed{-\frac{\varphi_2-\varphi_1}{L_2} + \frac{\varphi_3-\varphi_2}{L_3}}$$

$$\dot{Q}_3 = \boxed{-\frac{\varphi_3-\varphi_2}{L_3}}$$