

Q.1 Commutation Relation

Using the ladder operator expressions for the photon number operator and the single-mode electric field operator, show that

$$[\hat{n}, \hat{E}_x] = \epsilon_0 \sin(kz) (\hat{a}^\dagger - \hat{a}).$$

Solution:

The number operator and the single-mode cavity electric field are expressed via ladder operators as

$$\hat{n} = \hat{a}^\dagger \hat{a}, \quad \hat{E}_x = \epsilon_0 \sin(kz) (\hat{a} + \hat{a}^\dagger).$$

Expanding the commutator:

$$[\hat{n}, \hat{E}_x] = \epsilon_0 \sin(kz) [\hat{a}^\dagger \hat{a}, \hat{a} + \hat{a}^\dagger] = \epsilon_0 \sin(kz) \left([\hat{a}^\dagger \hat{a}, \hat{a}] + [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] \right).$$

We evaluate each commutator using the identity $[AB, C] = A[B, C] + [A, C]B$ together with $[\hat{a}, \hat{a}^\dagger] = 1$:

First commutator:

$$[\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}] \hat{a} = 0 + (-1) \hat{a} = -\hat{a}.$$

Second commutator:

$$[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger] \hat{a} = \hat{a}^\dagger (1) + 0 = \hat{a}^\dagger.$$

Substituting back:

$$[\hat{n}, \hat{E}_x] = \epsilon_0 \sin(kz) (-\hat{a} + \hat{a}^\dagger) = \boxed{\epsilon_0 \sin(kz) (\hat{a}^\dagger - \hat{a})}.$$

Q.2 Matrix Representation

(a) Matrix form of \hat{a} and \hat{a}^\dagger

Express the ladder operators in matrix form using their action on Fock states.

Solution:

The ladder operators act on Fock states as

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle.$$

Truncating to the two-level (qubit) subspace $\{|0\rangle, |1\rangle\}$ with basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

Annihilation operator: $\hat{a} |0\rangle = 0$, $\hat{a} |1\rangle = |0\rangle$:

$$\hat{a} = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Creation operator: $\hat{a}^\dagger |0\rangle = |1\rangle$, $\hat{a}^\dagger |1\rangle \rightarrow 0$ (truncated):

$$\hat{a}^\dagger = |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(b) Verification of identities

For identities 1–2 we compute directly with the 2×2 matrices from part (a). For identities 3–6 we normal-order in the full Fock space using $[\hat{a}, \hat{a}^\dagger] = 1$, then truncate to $\{|0\rangle, |1\rangle\}$: any monomial $(\hat{a}^\dagger)^m \hat{a}^n$ with $m \geq 2$ or $n \geq 2$ vanishes, and $\hat{n} \rightarrow \frac{I - \sigma_z}{2}$. The non-trivial Pauli component is then extracted. See also `q2_matrix_rep.ipynb` for numerical verification.

Solution:**Identity 1:** $\hat{a}^\dagger - \hat{a} = -i\sigma_y$

$$\hat{a}^\dagger - \hat{a} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \boxed{-i\sigma_y} \checkmark$$

Identity 2: $\hat{a}^\dagger + \hat{a} = \sigma_x$

$$\hat{a}^\dagger + \hat{a} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boxed{\sigma_x} \checkmark$$

Identity 3: $(\hat{a}^\dagger - \hat{a})^2$ Normal-ordering: $(\hat{a}^\dagger - \hat{a})^2 = (\hat{a}^\dagger)^2 + \hat{a}^2 - 2\hat{a}^\dagger \hat{a} - 1$. After truncation:

$$-2\hat{n} - I = -2 \cdot \frac{I - \sigma_z}{2} - I = -2I + \sigma_z \implies \boxed{+\sigma_z} \checkmark$$

Note: The problem states $-\sigma_z$; however the calculation gives $+\sigma_z$, indicating a likely sign error in the problem sheet.**Identity 4:** $(\hat{a}^\dagger + \hat{a})^2 = -\sigma_z$ Normal-ordering: $(\hat{a}^\dagger + \hat{a})^2 = (\hat{a}^\dagger)^2 + \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + 1$. After truncation:

$$2\hat{n} + I = 2 \cdot \frac{I - \sigma_z}{2} + I = 2I - \sigma_z \implies \boxed{-\sigma_z} \checkmark$$

Identity 5: $(\hat{a}^\dagger + \hat{a})^3 = 3\sigma_x$ Multiplying the normal-ordered square by $(\hat{a}^\dagger + \hat{a})$ and normal-ordering:

$$(\hat{a}^\dagger + \hat{a})^3 = (\hat{a}^\dagger)^3 + 3(\hat{a}^\dagger)^2 \hat{a} + 3\hat{a}^\dagger \hat{a}^2 + \hat{a}^3 + 3\hat{a}^\dagger + 3\hat{a}.$$

Only linear terms survive truncation: $3(\hat{a}^\dagger + \hat{a}) = 3\sigma_x$ by Identity 2.

$$\boxed{3\sigma_x} \checkmark$$

Identity 6: $(\hat{a}^\dagger + \hat{a})^4 = -6\sigma_z$ Multiplying the normal-ordered cube by $(\hat{a}^\dagger + \hat{a})$ and normal-ordering:

$$(\hat{a}^\dagger + \hat{a})^4 = (\hat{a}^\dagger)^4 + 4(\hat{a}^\dagger)^3 \hat{a} + 6(\hat{a}^\dagger)^2 \hat{a}^2 + 4\hat{a}^\dagger \hat{a}^3 + \hat{a}^4 + 6(\hat{a}^\dagger)^2 + 6\hat{a}^2 + 12\hat{n} + 3.$$

Only $12\hat{n} + 3$ survives truncation:

$$12 \cdot \frac{I - \sigma_z}{2} + 3I = 9I - 6\sigma_z \implies \boxed{-6\sigma_z} \checkmark$$

Two-mode identities (7–10). For independent modes $i \neq j$, operators on different modes commute ($[\hat{a}_i, \hat{a}_j^\dagger] = 0$), so each factor reduces by the single-mode results above.**Identity 7:** $(\hat{a}_i^\dagger - \hat{a}_i)(\hat{a}_j^\dagger - \hat{a}_j) = -\sigma_{iy}\sigma_{jy}$

By Identity 1 applied to each mode:

$$(-i\sigma_{iy})(-i\sigma_{jy}) = (-i)^2 \sigma_{iy}\sigma_{jy} = \boxed{-\sigma_{iy}\sigma_{jy}} \checkmark$$

Identity 8: $(\hat{a}_i^\dagger + \hat{a}_i)(\hat{a}_j^\dagger + \hat{a}_j) = \sigma_{ix}\sigma_{jx}$

By Identity 2 applied to each mode:

$$\sigma_{ix} \cdot \sigma_{jx} = \boxed{\sigma_{ix}\sigma_{jx}} \checkmark$$

Identity 9: $(\hat{a}_i^\dagger + \hat{a}_i)^3(\hat{a}_j^\dagger + \hat{a}_j) = 3\sigma_{ix}\sigma_{jx}$ By Identity 5 on mode i and Identity 2 on mode j :

$$3\sigma_{ix} \cdot \sigma_{jx} = \boxed{3\sigma_{ix}\sigma_{jx}} \checkmark$$

Identity 10: $(\hat{a}_i^\dagger + \hat{a}_i)^2(\hat{a}_j^\dagger + \hat{a}_j)^2 = \sigma_{iz}\sigma_{jz} - 2\sigma_{iz} - 2\sigma_{jz}$ From Identity 4's full computation, $(\hat{a}^\dagger + \hat{a})^2 \rightarrow 2I - \sigma_z$ in the qubit subspace. Taking the product over two modes:

$$(2I_i - \sigma_{iz})(2I_j - \sigma_{jz}) = 4I - 2\sigma_{iz} - 2\sigma_{jz} + \sigma_{iz}\sigma_{jz}.$$

Extracting the Pauli component (dropping the $4I$ identity term):

$$\boxed{\sigma_{iz}\sigma_{jz} - 2\sigma_{iz} - 2\sigma_{jz}} \checkmark$$

Q.3 Problem 2.5

Suppose the state of a single-mode cavity field is given at time $t = 0$ by

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + e^{i\phi}|n+1\rangle),$$

where ϕ is some phase. Find the state $|\psi(t)\rangle$ at times $t > 0$. For this time-evolved state, verify the uncertainty relation

$$\Delta n \Delta E_x \geq \frac{1}{2} \epsilon_0 |\sin(kz)| |\langle \hat{a}^\dagger - \hat{a} \rangle|.$$

Solution:

(i) **Time evolution.** The Hamiltonian $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})$ gives $e^{-i\hat{H}t/\hbar}|m\rangle = e^{-i\omega(m+1/2)t}|m\rangle$, so

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\omega(n+\frac{1}{2})t} |n\rangle + e^{i\phi} e^{-i\omega(n+\frac{3}{2})t} |n+1\rangle \right) = \frac{e^{-i\omega(n+\frac{1}{2})t}}{\sqrt{2}} \left(|n\rangle + e^{i(\phi-\omega t)} |n+1\rangle \right).$$

Up to a global phase:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|n\rangle + e^{i\theta} |n+1\rangle \right), \quad \theta \equiv \phi - \omega t.$$

(ii) **Photon-number uncertainty.**

$$\langle \hat{n} \rangle = n + \frac{1}{2}, \quad \langle \hat{n}^2 \rangle = \frac{1}{2}(n^2 + (n+1)^2) = n^2 + n + \frac{1}{2}.$$

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = (n^2 + n + \frac{1}{2}) - (n + \frac{1}{2})^2 = \frac{1}{4} \implies \Delta n = \frac{1}{2}.$$

(iii) **Field-operator expectation values.**

Only cross-terms between $|n\rangle$ and $|n+1\rangle$ contribute:

$$\langle \hat{a} \rangle = \frac{\sqrt{n+1}}{2} e^{i\theta}, \quad \langle \hat{a}^\dagger \rangle = \frac{\sqrt{n+1}}{2} e^{-i\theta}.$$

$$\langle \hat{a}^\dagger + \hat{a} \rangle = \sqrt{n+1} \cos \theta, \quad \langle \hat{a}^\dagger - \hat{a} \rangle = -i\sqrt{n+1} \sin \theta.$$

Since \hat{a}^2 and $(\hat{a}^\dagger)^2$ shift the Fock index by ± 2 , their expectation values vanish, giving $\langle (\hat{a} + \hat{a}^\dagger)^2 \rangle = 2\langle \hat{n} \rangle + 1 = 2n + 2$.

(iv) **Electric-field uncertainty.**

With $\hat{E}_x = \epsilon_0 \sin(kz) (\hat{a} + \hat{a}^\dagger)$:

$$(\Delta E_x)^2 = \epsilon_0^2 \sin^2(kz) [(2n+2) - (n+1) \cos^2 \theta] = \epsilon_0^2 \sin^2(kz) (n+1)(1 + \sin^2 \theta).$$

$$\Delta E_x = |\epsilon_0 \sin(kz)| \sqrt{(n+1)(1 + \sin^2 \theta)}.$$

(v) **Verification of the uncertainty relation.**

From Q.1, $[\hat{n}, \hat{E}_x] = \epsilon_0 \sin(kz) (\hat{a}^\dagger - \hat{a})$, so the uncertainty principle requires $\Delta n \Delta E_x \geq \frac{1}{2} |\epsilon_0 \sin(kz)| |\langle \hat{a}^\dagger - \hat{a} \rangle|$. Substituting:

$$\text{LHS} = \frac{1}{2} |\epsilon_0 \sin(kz)| \sqrt{(n+1)(1 + \sin^2 \theta)},$$

$$\text{RHS} = \frac{1}{2} |\epsilon_0 \sin(kz)| \sqrt{n+1} |\sin \theta|.$$

Dividing by the common factor $\frac{1}{2} |\epsilon_0 \sin(kz)| \sqrt{n+1}$, the inequality reduces to $\sqrt{1 + \sin^2 \theta} \geq |\sin \theta|$, which holds trivially since $1 + \sin^2 \theta \geq \sin^2 \theta$.

$$\Delta n \Delta E_x = \frac{1}{2} |\epsilon_0 \sin(kz)| \sqrt{(n+1)(1 + \sin^2 \theta)} \geq \frac{1}{2} |\epsilon_0 \sin(kz)| |\langle \hat{a}^\dagger - \hat{a} \rangle|. \quad \checkmark$$

Q.4 Problem 2.7

Many processes involve the absorption of single photons from a quantum field state, the process of absorption being represented by the action of the annihilation operator \hat{a} . For an arbitrary field state $|\psi\rangle$, the absorption of a single photon yields the state $|\psi'\rangle \sim \hat{a}|\psi\rangle$. Normalize this state. Compare the average photon numbers \bar{n} of $|\psi\rangle$ and \bar{n}' of $|\psi'\rangle$. Do you find that $\bar{n}' = \bar{n} - 1$?

Solution:

(i) **Normalization.** The norm of $\hat{a}|\psi\rangle$ is $\|\hat{a}|\psi\rangle\|^2 = \langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = \bar{n}$, so

$$|\psi'\rangle = \frac{\hat{a}|\psi\rangle}{\sqrt{\bar{n}}}.$$

(ii) **Average photon number after absorption.**

$$\bar{n}' = \langle\psi'|\hat{n}|\psi'\rangle = \frac{\langle\psi|\hat{a}^\dagger\hat{n}\hat{a}|\psi\rangle}{\bar{n}}.$$

Using $[\hat{n}, \hat{a}] = -\hat{a}$ gives $\hat{a}^\dagger\hat{n}\hat{a} = \hat{n}^2 - \hat{n}$, so

$$\bar{n}' = \frac{\langle\hat{n}^2\rangle - \bar{n}}{\bar{n}} = \frac{\bar{n}^2 + (\Delta n)^2 - \bar{n}}{\bar{n}} \implies \bar{n}' = \bar{n} - 1 + \frac{(\Delta n)^2}{\bar{n}}.$$

(iii) **Is $\bar{n}' = \bar{n} - 1$?**

No. Equality holds only when $(\Delta n)^2 = 0$, i.e. for Fock states. Otherwise $\bar{n}' > \bar{n} - 1$ because \hat{a} acts with amplitude $\sqrt{\bar{n}}$, preferentially selecting higher photon-number components.

Q.5 Problem 2.8

Consider the superposition of the vacuum and 10-photon number state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle).$$

Calculate the average photon number for this state. Next assume that a single photon is absorbed and recalculate the average photon number. Does your result seem sensible in comparison with your answer to the previous question?

Solution:

(i) **Average photon number before absorption.**

$$\bar{n} = \langle\psi|\hat{n}|\psi\rangle = \frac{1}{2}(0 + 10) = \boxed{5}.$$

(ii) **State after single-photon absorption.**

Since $\hat{a}|0\rangle = 0$ and $\hat{a}|10\rangle = \sqrt{10}|9\rangle$:

$$\hat{a}|\psi\rangle = \frac{\sqrt{10}}{\sqrt{2}}|9\rangle \implies |\psi'\rangle = \frac{\hat{a}|\psi\rangle}{\sqrt{\bar{n}}} = \frac{\sqrt{10}/\sqrt{2}}{\sqrt{5}}|9\rangle = |9\rangle.$$

(iii) **Average photon number after absorption.**

$$\bar{n}' = \langle 9|\hat{n}|9\rangle = \boxed{9}.$$

(iv) **Comparison with Q.4.**

Verifying with the general formula: $(\Delta n)^2 = \frac{1}{2}(0^2 + 10^2) - 25 = 25$, so

$$\bar{n}' = 5 - 1 + \frac{25}{5} = 9.\checkmark$$

Rather than the naive $\bar{n} - 1 = 4$, we get $\bar{n}' = 9$ because \hat{a} annihilates the vacuum component entirely ($\hat{a}|0\rangle = 0$) while acting with amplitude $\sqrt{10}$ on $|10\rangle$, collapsing the superposition onto $|9\rangle$.