Assignment – 04 - PHYS 514

Due Monday Feb 15 – 12:00 mid night

(Late submission will not be accepted)

Submit Question # 2, 3, 6 and 7 ONLY

You can submit the hard copy or soft-copy via email – please make to write your name and ID

Q.1 Expectation Value of Position ϕ

Show that for Fock state $|n\rangle$

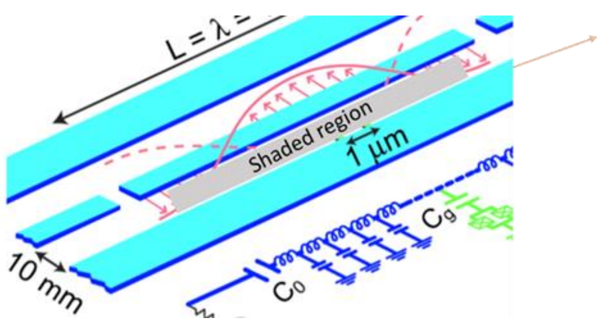
$$\langle\hat{\phi}\rangle = 0$$

$$\langle\hat{\phi}^2\rangle = (2n+1)\Phi_{ZPF}^2$$

Q.2 Zero Point Fluctuation (ZPF)

Calculate the zero point fluctuation in Voltage V_{ZPF} for an LC oscillator circuit where L= 50 nH and C = 70 pF.

Further, Assume the LC oscillator (the resonator) in following Fig is in vacuum (zero photon) state. Find the average electric field, E in the shaded region



Q.3 Coherent State

A coherent state, $|\alpha\rangle$, is the eigenstate of operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

(a) coherent state, $|\alpha\rangle$is the eigenstate of $\hat{a}^\dagger$as well (Y/N)

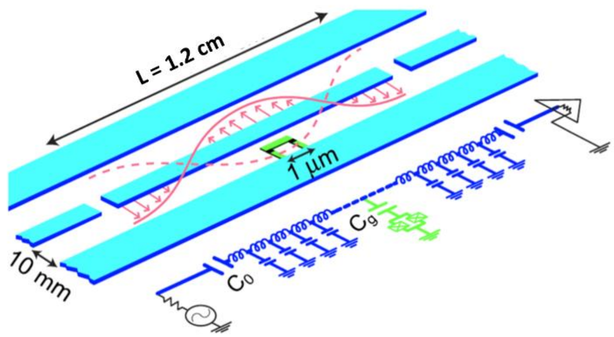
(b) $\hat{a}$ is a lowering operator, briefly explain how the coherent state $|\alpha\rangle$ can be an eigenstate of operator $\hat{a}$

Q.4 Resonator

Consider a coplanar waveguide cavity formed between two coupling capacitors, a central conducting strip (shown in Fig below).

(a) If length of central strip is 1.2 cm, find the frequency of *given harmonic* and first harmonic.

(b) Find the energy of first harmonic



Q.5 Superconductivity

Briefly describe the following in reference to superconductors

(i) Zero resistance

(ii) Two-fluid behavior

(iii) Energy gap

(iv) Meissner effect

(v) Long Range Coherence

Q.6 Coherent State

A coherent state, $|\alpha\rangle$, is a special quantum state of a resonator, which is very useful in quantum optics and circuit QED, because it behaves almost classically. It can be defined to be the eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

(c) Starting with this definition, show that the coherent state can be written in the number-state basis as:

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

(*Hint: use following step i) and ii)*

i) Writing the coherent state $|\alpha\rangle$, as an arbitrary superposition of number states

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

and substitute this expression into the eigenstate equation above to derive the recursion relation for the coefficients

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

Hence show that

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0 \, |n\rangle$$

ii) Use the normalisation of the coherent state, $\langle\alpha|\alpha\rangle = 1$, to show that:

$$c_0 = e^{-\frac{1}{2}|\alpha|^2}$$

(d) Using the definition of the coherent state, show that the average excitation number

$$\langle\hat{n}\rangle = |\alpha|^2$$

(e) Starting again from the definition of the coherent state, and using the result of the previous question, show that the variance (width) of the excitation-number distribution is also

$$\Delta\hat{n} = |\alpha|^2$$

(*Remember:* $\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$)

(f) The goal of this question is to study how a coherent state evolves under the oscillator's free evolution Hamiltonian. Firstly, given an initial Fock state, $|\psi(0)\rangle = |n\rangle$, show that:

$$|\psi(t)\rangle = e^{in\omega_0 t}|n\rangle$$

[Hint: Use the operator form of the Taylor expansion for an exponential,

$$e^{\hat{A}} = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$use\ \hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} \quad instead\ of \quad \hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$$

(g) Secondly, given a coherent state as initial state, $|\psi(0)\rangle = |\alpha\rangle$, use this previous result in conjunction with the number-state decomposition from (a) to show that:

$$|\psi(t)\rangle = |\alpha e^{i\omega_0 t}\rangle \equiv |\alpha(t)\rangle$$

Note: This result underpins the "classical" behavior of the coherent state, because its form and evolution are both completely described a single classical complex number.

Q.7 Phase Space of an Oscillator

A complex number, α (in reference to coherent state, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ can be described in terms of an amplitude and phase, via

$$\alpha = |\alpha| e^{i\theta}$$

where real and imaginary parts of α are

$$Re\ \{\alpha\} = \frac{1}{2}(\alpha + \alpha^*) = |\alpha|cos\theta$$
$$Im\ \alpha\} = -\frac{i}{2}(\alpha - \alpha^*) = |\alpha|sin\theta$$

We can define new operators for real and imagery part of amplitude

$$\hat{X}_1 = Re\ \{\alpha\} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$
$$\hat{X}_2 = Im\ \{\alpha\} = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger)$$

These new operators now represent the phase-space coordinates for the quantum field, often called the "quadrature" operators (for historical reasons). They also define the coordinates of a 2D pictorial phase-space representation of the quantum oscillator mode.

(a) Show that
$$[\hat{X}_1, \hat{X}_2\ ] = \frac{i}{2}$$
(b) Consider a system in the vacuum or ground state, $|\psi\rangle = |0\rangle|$, show that
$$\langle\hat{X}_1\rangle = \langle\hat{X}_2\rangle = 0$$
$$\Delta\hat{X}_1^2 = \Delta\hat{X}_2^2 = \frac{1}{4}$$
(c) Consider a system in the Fock state, $|\psi\rangle = |n\rangle|$, show that
$$\langle\hat{X}_1\rangle = \langle\hat{X}_2\rangle = 0$$
$$\Delta\hat{X}_1^2 = \Delta\hat{X}_2^2 = \frac{n}{2} + \frac{1}{4}$$

(d) Consider a system in the Coherent state, $|\psi\rangle = |\alpha\rangle|$, show that
$$\langle\hat{X}_1\rangle = |\alpha|cos\theta$$
$$\langle\hat{X}_2\rangle = |\alpha|sin\theta$$
$$\Delta\hat{X}_1^2 = \Delta\hat{X}_2^2 = \frac{1}{4}$$

Note: These results highlight the different, complementary nature of the number states and coherent states.