

Q.2 Zero Point Fluctuation (ZPF)

$L = 50 \text{ nH}$, $C = 70 \text{ pF}$, gap $d = 1 \mu\text{m}$.

Resonance frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50 \times 10^{-9} \times 70 \times 10^{-12}}} = \frac{1}{\sqrt{3.5 \times 10^{-18}}} \approx 5.345 \times 10^8 \text{ rad/s}$$

V_{ZPF} :

The LC Hamiltonian $\hat{H} = \hat{Q}^2/2C + \hat{\Phi}^2/2L$ maps to a harmonic oscillator with mass C . Zero-point charge fluctuation $Q_{\text{ZPF}} = \sqrt{\hbar C \omega_0/2}$, so $V_{\text{ZPF}} = Q_{\text{ZPF}}/C$:

$$V_{\text{ZPF}} = \sqrt{\frac{\hbar \omega_0}{2C}} = \sqrt{\frac{1.055 \times 10^{-34} \times 5.345 \times 10^8}{2 \times 70 \times 10^{-12}}} = \sqrt{4.026 \times 10^{-16}} \approx \boxed{20 \text{ nV}}$$

Electric field in shaded region (vacuum state):

In $|0\rangle$, $\langle \hat{V} \rangle = 0$, so $\langle E \rangle = 0$. The ZPF electric field across the gap is:

$$E_{\text{ZPF}} = \frac{V_{\text{ZPF}}}{d} = \frac{20 \times 10^{-9}}{1 \times 10^{-6}} \approx \boxed{20 \text{ mV/m}}$$

Q.3 Coherent State

(a) Is $|\alpha\rangle$ an eigenstate of $\hat{a}^\dagger$? No

Suppose $|\alpha\rangle$ is an eigenstate of both $\hat{a}$ and $\hat{a}^\dagger$ with eigenvalues α and β . Then:

$$[\hat{a}, \hat{a}^\dagger]|\alpha\rangle = \hat{a}\hat{a}^\dagger|\alpha\rangle - \hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\beta|\alpha\rangle - \beta\alpha|\alpha\rangle = 0$$

But $[\hat{a}, \hat{a}^\dagger] = 1$ requires $[\hat{a}, \hat{a}^\dagger]|\alpha\rangle = |\alpha\rangle \neq 0$. Contradiction. $\square$

(b) How can $|\alpha\rangle$ be an eigenstate of $\hat{a}$?

A Fock state $|n\rangle$ maps to a different state $\sqrt{n}|n-1\rangle$ under $\hat{a}$. A coherent state is an infinite superposition over all $|n\rangle$ with Poissonian coefficients satisfying $c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n$. Applying $\hat{a}$ shifts each term $|n\rangle \rightarrow \sqrt{n}|n-1\rangle$; the recursion ensures the resulting sum equals $\alpha|\alpha\rangle$ rather than a different state.

Q.6 Coherent State

(c) Number-state decomposition

Write $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ and substitute into $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$:

$$\sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle \quad \xrightarrow{\text{reindex}} \quad c_{n+1} \sqrt{n+1} = \alpha c_n \quad \Rightarrow \quad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

Normalisation $\langle\alpha|\alpha\rangle = 1$:

$$|c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} = 1 \quad \Rightarrow \quad c_0 = e^{-\frac{1}{2}|\alpha|^2}$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

(d) Average excitation number

Using $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$:

$$\langle\hat{n}\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha^*\alpha\langle\alpha|\alpha\rangle \quad \Rightarrow \quad \boxed{\langle\hat{n}\rangle = |\alpha|^2}$$

(e) Variance of excitation number

Using $\hat{a}\hat{a}^\dagger = \hat{n} + 1$:

$$\begin{aligned} \hat{n}^2 &= \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} = \hat{a}^\dagger(\hat{n} + 1)\hat{a} = (\hat{a}^\dagger)^2\hat{a}^2 + \hat{n} \\ \langle\hat{n}^2\rangle &= |\alpha|^4 + |\alpha|^2 \quad \Rightarrow \quad \text{Var}(\hat{n}) = |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2 \quad \Rightarrow \quad \boxed{\Delta\hat{n} = |\alpha|} \end{aligned}$$

(f) Time evolution of Fock state

$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|n\rangle = e^{-i\omega_0\hat{n}t}|n\rangle$. Since $\hat{n}^k|n\rangle = n^k|n\rangle$:

$$e^{-i\omega_0\hat{n}t}|n\rangle = \sum_{k=0}^{\infty} \frac{(-i\omega_0t)^k}{k!} n^k |n\rangle = e^{-in\omega_0t}|n\rangle \quad \Rightarrow \quad \boxed{|\psi(t)\rangle = e^{-in\omega_0t}|n\rangle}$$

(g) Time evolution of coherent state

Applying the result of (f) to the decomposition from (c):

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega_0t}|n\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{(\alpha e^{-i\omega_0t})^n}{\sqrt{n!}} |n\rangle \quad \Rightarrow \quad \boxed{|\psi(t)\rangle = |\alpha e^{-i\omega_0t}\rangle}$$

The coherent state remains coherent; α simply rotates in the complex plane at ω_0 .

Q.7 Phase Space of an Oscillator

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger)$$

(a) **Commutator** $[\hat{X}_1, \hat{X}_2]$

$$[\hat{X}_1, \hat{X}_2] = \frac{1}{2} \cdot \left(-\frac{i}{2}\right) [\hat{a} + \hat{a}^\dagger, \hat{a} - \hat{a}^\dagger] = -\frac{i}{4} (0 - 1 + (-1) - 0) = -\frac{i}{4}(-2) = \boxed{\frac{i}{2}}$$

(b) **Vacuum state** $|0\rangle$

$$\hat{a}|0\rangle = 0, \text{ so } \langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0.$$

For the variance, expanding $(\hat{a} + \hat{a}^\dagger)^2$, only $\hat{a}\hat{a}^\dagger = \hat{n} + 1$ survives in $|0\rangle$:

$$\langle \hat{X}_1^2 \rangle = \frac{1}{4} \langle 0 | (\hat{a} + \hat{a}^\dagger)^2 | 0 \rangle = \frac{1}{4} (1) = \frac{1}{4}$$

Similarly $\langle 0 | (\hat{a} - \hat{a}^\dagger)^2 | 0 \rangle = -1$, so $\langle \hat{X}_2^2 \rangle = (-\frac{i}{2})^2 (-1) = \frac{1}{4}$.

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0, \quad \Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{1}{4}$$

(c) **Fock state** $|n\rangle$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ are both orthogonal to $|n\rangle$, so $\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0$.

The $\hat{a}^2$ and $(\hat{a}^\dagger)^2$ terms in $(\hat{a} + \hat{a}^\dagger)^2$ connect to $|n \pm 2\rangle$ and vanish:

$$\langle n | (\hat{a} + \hat{a}^\dagger)^2 | n \rangle = \langle n | \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} | n \rangle = (n+1) + n = 2n+1$$

$$\langle \hat{X}_1^2 \rangle = \frac{2n+1}{4} \Rightarrow \Delta \hat{X}_1^2 = \frac{2n+1}{4} = \frac{n}{2} + \frac{1}{4}$$

By identical steps $\Delta \hat{X}_2^2 = \frac{n}{2} + \frac{1}{4}$.

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0, \quad \Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{n}{2} + \frac{1}{4}$$

(d) **Coherent state** $|\alpha\rangle$, $\alpha = |\alpha|e^{i\theta}$

$$\langle \hat{X}_1 \rangle = \frac{1}{2}(\alpha + \alpha^*) = |\alpha| \cos \theta, \quad \langle \hat{X}_2 \rangle = -\frac{i}{2}(\alpha - \alpha^*) = -\frac{i}{2}(2i|\alpha| \sin \theta) = |\alpha| \sin \theta$$

For $\Delta \hat{X}_1^2$, using $\langle \hat{a}^2 \rangle = \alpha^2$, $\langle \hat{a}\hat{a}^\dagger \rangle = |\alpha|^2 + 1$, $\langle \hat{a}^\dagger\hat{a} \rangle = |\alpha|^2$, $\langle (\hat{a}^\dagger)^2 \rangle = (\alpha^*)^2$:

$$\langle \hat{X}_1^2 \rangle = \frac{1}{4}(\alpha^2 + |\alpha|^2 + 1 + |\alpha|^2 + (\alpha^*)^2) = \frac{1}{4}((\alpha + \alpha^*)^2 + 1) = |\alpha|^2 \cos^2 \theta + \frac{1}{4}$$

$$\Delta \hat{X}_1^2 = |\alpha|^2 \cos^2 \theta + \frac{1}{4} - |\alpha|^2 \cos^2 \theta = \frac{1}{4}$$

For $\hat{X}_2$: $\langle \hat{X}_2^2 \rangle = (-\frac{i}{2})^2(\alpha^2 - (|\alpha|^2 + 1) - |\alpha|^2 + (\alpha^*)^2) = -\frac{1}{4}((\alpha - \alpha^*)^2 - 1) = |\alpha|^2 \sin^2 \theta + \frac{1}{4}$

$$\Delta \hat{X}_2^2 = \frac{1}{4}$$

$$\langle \hat{X}_1 \rangle = |\alpha| \cos \theta, \quad \langle \hat{X}_2 \rangle = |\alpha| \sin \theta, \quad \Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{1}{4}$$