

Q.1 — Dressed states of the cavity–qubit system

The dressed states are

$$|n, -\rangle = \cos(\theta_n) |g\rangle |n+1\rangle - \sin(\theta_n) |e\rangle |n\rangle, \quad |n, +\rangle = \sin(\theta_n) |g\rangle |n+1\rangle + \cos(\theta_n) |e\rangle |n\rangle,$$

with the mixing angle obeying $\tan(2\theta_n) = \frac{2g\sqrt{n+1}}{\Delta}$, $\Delta = \omega_q - \omega_r$.

Solution:

(a) Eigenstates at $\omega_q = \omega_c$ (resonance, $\Delta = 0$):

At resonance $\tan(2\theta_n) \rightarrow \infty$, so $\theta_n = \pi/4$ and $\cos\theta_n = \sin\theta_n = 1/\sqrt{2}$.

$$|n, \pm\rangle = \frac{1}{\sqrt{2}} (|g, n+1\rangle \pm |e, n\rangle)$$

(b) Energies at $\omega_q = \omega_c \equiv \omega$:

Diagonalise $\hat{H}_{JC}$ in the $(n+1)$ -excitation manifold $M_n = \text{span}\{|g, n+1\rangle, |e, n\rangle\}$.

Diagonal entries ($\hat{H}_{\text{sym}}$). Using $\hat{\sigma}_z |g\rangle = +|g\rangle$ and qubit term $-\frac{\hbar\omega_q}{2}\hat{\sigma}_z$:

$$\langle g, n+1 | \hat{H}_{JC} |g, n+1\rangle = (n+1)\hbar\omega_r - \frac{\hbar\omega_q}{2}, \quad \langle e, n | \hat{H}_{JC} |e, n\rangle = n\hbar\omega_r + \frac{\hbar\omega_q}{2}.$$

(The interaction term has zero diagonal because $\hat{a}\hat{\sigma}^\dagger |g, n+1\rangle = \sqrt{n+1} |e, n\rangle \perp |g, n+1\rangle$, etc.)

Off-diagonal entry ($\hat{H}_{\text{off-diag}}$). The photon and qubit terms vanish (orthogonal qubit states for any σ_z -diagonal operator); only the interaction contributes:

$$\langle e, n | \hat{H}_{JC} |g, n+1\rangle = \hbar g \langle e, n | \hat{a}\hat{\sigma}^\dagger |g, n+1\rangle = \hbar g \sqrt{n+1}.$$

Effective 2×2 matrix:

$$\hat{H}_{JC}|_{M_n} = \begin{pmatrix} (n+1)\hbar\omega_r - \frac{\hbar\omega_q}{2} & \hbar g \sqrt{n+1} \\ \hbar g \sqrt{n+1} & n\hbar\omega_r + \frac{\hbar\omega_q}{2} \end{pmatrix}.$$

Mean $= \hbar\omega_r(n + \frac{1}{2})$ (the ω_q contributions cancel).

Half-difference $= -\hbar\Delta/2$ with $\Delta = \omega_q - \omega_r$.

The eigenvalues are

$$E_{n,\pm} = \hbar\omega_r(n + \frac{1}{2}) \pm \frac{\hbar}{2} \sqrt{\Delta^2 + 4g^2(n+1)}.$$

At resonance $\omega_q = \omega_c \equiv \omega$ ($\Delta = 0$):

$$E_{n,\pm} = \hbar\omega(n + \frac{1}{2}) \pm \hbar g \sqrt{n+1}$$

(c) Condition for $\theta_n = 0$:

From $\tan(2\theta_n) = 2g\sqrt{n+1}/\Delta$, $\theta_n \rightarrow 0$ requires the **large detuning (dispersive) limit**

$$|\Delta| = |\omega_q - \omega_r| \gg g\sqrt{n+1}.$$

In this limit the dressed states collapse onto the bare product states:

$$|n, -\rangle \longrightarrow |g, n+1\rangle, \quad |n, +\rangle \longrightarrow |e, n\rangle$$

Experimentally this is realised by detuning the qubit far from the cavity.

Q.2 — Quality factor, Rabi vs. Ramsey, T_1 and T_2

Solution:

(a) Quality factor of cavity / CPW.

Q is the ratio of energy stored to energy lost per radian:

$$Q = \omega_0 \frac{(\text{energy stored})}{(\text{power dissipated})} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f_{\text{FWHM}}}.$$

Spectral measurement: drive the resonator with a VNA and record $|S_{21}(f)|^2$ (transmission) or $|S_{11}(f)|^2$ (reflection). The response is a Lorentzian centred at f_0 ; fit it to extract f_0 and the full-width at half-maximum Δf . The (loaded) quality factor is then $Q_L = f_0/\Delta f$. Internal and external Q are separated by fitting the complex S_{21} .

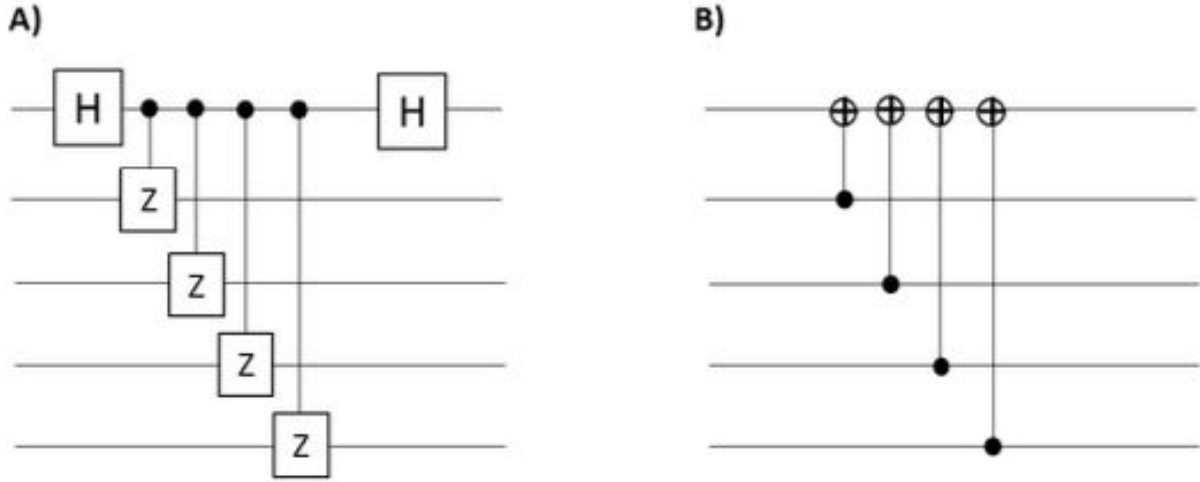
(b) Rabi vs. Ramsey.

- **Rabi:** a single resonant drive of variable duration t ; population $P_e(t) = \sin^2(\Omega_R t/2)$ oscillates between $|g\rangle$ and $|e\rangle$. It calibrates the drive amplitude and the $\pi/\pi/2$ pulse times and gives the Rabi frequency Ω_R .
- **Ramsey:** two $\pi/2$ pulses separated by a free-evolution time τ with a small detuning $\delta = \omega_d - \omega_q$; the population oscillates as $\propto \cos(\delta\tau)e^{-\tau/T_2^*}$. It measures the qubit frequency precisely and yields the pure dephasing time T_2^* .

(c) T_1 and T_2 .

- T_1 (energy relaxation): apply a π -pulse, wait a variable delay τ , then read out. $P_e(\tau) = e^{-\tau/T_1}$.
- T_2 (coherence/dephasing): obtained from Ramsey (T_2^*) or, removing low-frequency noise, from a Hahn-echo sequence $\pi/2 - \tau/2 - \pi - \tau/2 - \pi/2$ giving the echo time T_2 . They satisfy $\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_\varphi}$.

Q.3 — Equivalence of circuits A and B



Circuit A acts on qubit 1 as

$$U_A = H_1 (CZ_{12})(CZ_{13})(CZ_{14})(CZ_{15}) H_1,$$

while circuit B is the cascade

$$U_B = \text{CNOT}_{2 \rightarrow 1} \text{CNOT}_{3 \rightarrow 1} \text{CNOT}_{4 \rightarrow 1} \text{CNOT}_{5 \rightarrow 1}.$$

Solution:

Key identity. Because CZ is symmetric in its two qubits, conjugating it by Hadamards on the target line gives a CNOT:

$$H_1 CZ_{1j} H_1 = \text{CNOT}_{j \rightarrow 1}.$$

Mathematical proof. Each pair of Hadamards is the identity, $H_1 H_1 = I$. Insert the identity $I = H_1 H_1$ between every adjacent pair of CZ gates in circuit A:

$$\begin{aligned} U_A &= H_1 CZ_{12} CZ_{13} CZ_{14} CZ_{15} H_1 \\ &= H_1 CZ_{12} \underbrace{(H_1 H_1)}_I CZ_{13} \underbrace{(H_1 H_1)}_I CZ_{14} \underbrace{(H_1 H_1)}_I CZ_{15} H_1 \\ &= (H_1 CZ_{12} H_1) (H_1 CZ_{13} H_1) (H_1 CZ_{14} H_1) (H_1 CZ_{15} H_1) \\ &= \text{CNOT}_{2 \rightarrow 1} \text{CNOT}_{3 \rightarrow 1} \text{CNOT}_{4 \rightarrow 1} \text{CNOT}_{5 \rightarrow 1} \\ &= U_B. \end{aligned}$$

Hence

$$U_A = U_B$$

and the two circuits implement the same unitary. ■

Q.4 — JC Hamiltonian and the excitation operator

$$\hat{H}_{JC} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_q}{2} \hat{\sigma}_z + i\hbar g (\hat{a}\hat{\sigma}^\dagger - \hat{a}^\dagger \hat{\sigma}), \quad \hat{E} = \hat{a}^\dagger \hat{a} + \hat{\sigma}^\dagger \hat{\sigma}.$$

Solution:

(a) **Full commutator expansion** $[\hat{H}_{JC}, \hat{E}] = 0$.

Break $\hat{H}_{JC}$ into three pieces and use $\hat{E} = \hat{N}_a + \hat{N}_\sigma$ with $\hat{N}_a = \hat{a}^\dagger \hat{a}$, $\hat{N}_\sigma = \hat{\sigma}^\dagger \hat{\sigma}$.

Photon term:

$$[\hat{a}^\dagger \hat{a}, \hat{N}_a] = 0, \quad [\hat{a}^\dagger \hat{a}, \hat{N}_\sigma] = 0,$$

because $\hat{N}_\sigma$ acts only on the qubit subspace.

Qubit term: $\hat{\sigma}_z = 2\hat{\sigma}^\dagger \hat{\sigma} - \mathbf{1}$, so it is a function of $\hat{N}_\sigma$ alone, and

$$[\hat{\sigma}_z, \hat{N}_\sigma] = 0, \quad [\hat{\sigma}_z, \hat{N}_a] = 0.$$

Interaction term: use the standard relations

$$[\hat{a}, \hat{N}_a] = +\hat{a}, \quad [\hat{a}^\dagger, \hat{N}_a] = -\hat{a}^\dagger, \quad [\hat{\sigma}, \hat{N}_\sigma] = +\hat{\sigma}, \quad [\hat{\sigma}^\dagger, \hat{N}_\sigma] = -\hat{\sigma}^\dagger,$$

(easily checked: $[\hat{a}, \hat{a}^\dagger \hat{a}] = [\hat{a}, \hat{a}^\dagger] \hat{a} = \hat{a}$, and similarly for $\hat{\sigma}$ since $\hat{\sigma} \hat{\sigma}^\dagger \hat{\sigma} = \hat{\sigma}$, $\hat{\sigma}^\dagger \hat{\sigma} \hat{\sigma}^\dagger = \hat{\sigma}^\dagger$). Then

$$[\hat{a}\hat{\sigma}^\dagger, \hat{E}] = [\hat{a}, \hat{N}_a] \hat{\sigma}^\dagger + \hat{a} [\hat{\sigma}^\dagger, \hat{N}_\sigma] = (+\hat{a}) \hat{\sigma}^\dagger + \hat{a} (-\hat{\sigma}^\dagger) = 0,$$

$$[\hat{a}^\dagger \hat{\sigma}, \hat{E}] = [\hat{a}^\dagger, \hat{N}_a] \hat{\sigma} + \hat{a}^\dagger [\hat{\sigma}, \hat{N}_\sigma] = (-\hat{a}^\dagger) \hat{\sigma} + \hat{a}^\dagger (+\hat{\sigma}) = 0.$$

Putting it together,

$$[\hat{H}_{JC}, \hat{E}] = \hbar\omega_r \underbrace{[\hat{N}_a, \hat{E}]}_0 + \frac{\hbar\omega_q}{2} \underbrace{[\hat{\sigma}_z, \hat{E}]}_0 + i\hbar g \left(\underbrace{[\hat{a}\hat{\sigma}^\dagger, \hat{E}]}_0 - \underbrace{[\hat{a}^\dagger \hat{\sigma}, \hat{E}]}_0 \right) = \boxed{0}.$$

(b) Since $[\hat{H}_{JC}, \hat{E}] = 0$, the two operators **share a common eigenbasis** $\Rightarrow \boxed{Y}$.

(c) **Eigenstates of $\hat{E}$.** With $\hat{E} |n, s\rangle = (n + \delta_{s,e}) |n, s\rangle$:

$$\hat{E} |0, g\rangle = 0, \quad \hat{E} |n, g\rangle = n |n, g\rangle, \quad \hat{E} |n-1, e\rangle = n |n-1, e\rangle$$

Hence $|n, g\rangle$ and $|n-1, e\rangle$ are degenerate eigenvectors of $\hat{E}$ with eigenvalue n .

(d) **Action of $\hat{H}_{JC}$.**

Vacuum: $\hat{a} |0\rangle = 0$, $\hat{\sigma} |g\rangle = 0$, $\hat{\sigma}_z |g\rangle = -|g\rangle$, so

$$\hat{H}_{JC} |0, g\rangle = -\frac{\hbar\omega_q}{2} |0, g\rangle$$

General $|n, g\rangle$:

- $\hbar\omega_r \hat{a}^\dagger \hat{a} |n, g\rangle = n\hbar\omega_r |n, g\rangle$
- $\frac{\hbar\omega_q}{2} \hat{\sigma}_z |n, g\rangle = -\frac{\hbar\omega_q}{2} |n, g\rangle$
- $\hat{a}\hat{\sigma}^\dagger |n, g\rangle = \sqrt{n} |n-1, e\rangle$
- $\hat{a}^\dagger \hat{\sigma} |n, g\rangle = 0$

$$\hat{H}_{JC} |n, g\rangle = \left(n\hbar\omega_r - \frac{\hbar\omega_q}{2} \right) |n, g\rangle + i\hbar g \sqrt{n} |n-1, e\rangle$$

General $|n-1, e\rangle$:

- $\hbar\omega_r \hat{a}^\dagger \hat{a} |n-1, e\rangle = (n-1)\hbar\omega_r |n-1, e\rangle$
- $\frac{\hbar\omega_q}{2} \hat{\sigma}_z |n-1, e\rangle = +\frac{\hbar\omega_q}{2} |n-1, e\rangle$
- $\hat{a}\hat{\sigma}^\dagger |n-1, e\rangle = 0$
- $\hat{a}^\dagger \hat{\sigma} |n-1, e\rangle = \sqrt{n} |n, g\rangle$

$$\hat{H}_{JC} |n-1, e\rangle = \left((n-1)\hbar\omega_r + \frac{\hbar\omega_q}{2} \right) |n-1, e\rangle - i\hbar g \sqrt{n} |n, g\rangle$$

Note that within each n -excitation manifold $\{|n, g\rangle, |n-1, e\rangle\}$, $\hat{H}_{JC}$ acts as a 2×2 matrix — confirming the block structure implied by $[\hat{H}_{JC}, \hat{E}] = 0$.

(e) **Five possible states of the coupled cavity–qubit system:**

$$|0, g\rangle, |0, e\rangle, |1, g\rangle, |1, e\rangle, |2, g\rangle$$

Q.5 — JJ non-linear oscillator in the cooper-pair (charge) basis

$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\phi}. \quad (1)$$

Solution:

Note on the problem statement. The question writes “ $E_c = \frac{1}{2} \frac{(2e)^2}{C} = \frac{2e^2}{C}$ ”. With that definition, $\hat{Q}^2/(2C) = (2e\hat{n})^2/(2C) = 2e^2\hat{n}^2/C = E_c\hat{n}^2$, which gives $\hat{H} = E_c\hat{n}^2 - E_J \cos \hat{\phi}$, *not* $4E_c\hat{n}^2$. To obtain the standard transmon form $\hat{H} = 4E_c\hat{n}^2 - E_J \cos \hat{\phi}$ one must use the conventional single-electron charging energy

$$E_c \equiv \frac{e^2}{2C}$$

We adopt this standard definition throughout. The factor of 4 then arises because $\hat{n}$ counts *Cooper pairs* (charge $2e$), while E_c is defined per electron.

(a) Derivation of (2). Substitute $\hat{Q} = 2e\hat{n}$:

$$\frac{\hat{Q}^2}{2C} = \frac{(2e\hat{n})^2}{2C} = \frac{4e^2\hat{n}^2}{2C} = 4 \cdot \frac{e^2}{2C} \hat{n}^2 = 4E_c \hat{n}^2.$$

Therefore

$$\hat{H} = 4E_c \hat{n}^2 - E_J \cos \hat{\phi}. \quad (2)$$

(b) Matrix form in the cooper-pair number basis.

Using $\hat{n}|n\rangle = n|n\rangle$ and $\cos \hat{\phi} = \frac{1}{2}(e^{i\hat{\phi}} + e^{-i\hat{\phi}})$ with $e^{\pm i\hat{\phi}}|n\rangle = |n \pm 1\rangle$:

$$\begin{aligned} H_{mn} &= \langle m | \hat{H} | n \rangle = 4E_c n^2 \langle m | n \rangle - \frac{E_J}{2} (\langle m | n+1 \rangle + \langle m | n-1 \rangle) \\ &= 4E_c n^2 \delta_{m,n} - \frac{E_J}{2} (\delta_{m,n+1} + \delta_{m,n-1}). \end{aligned}$$

$$H_{mn} = 4E_c n^2 \delta_{m,n} - \frac{E_J}{2} (\delta_{m,n+1} + \delta_{m,n-1})$$

Symbolic non-zero entries (six explicit):

$$H_{00} = 0, \quad H_{11} = 4E_c, \quad H_{22} = 16E_c, \quad H_{01} = H_{10} = -\frac{E_J}{2}, \quad H_{12} = H_{21} = -\frac{E_J}{2}.$$

(Diagonals continue as $H_{33} = 36E_c$, $H_{44} = 64E_c$, ... and off-diagonals as $H_{23} = H_{32} = -E_J/2$, etc.)

Explicit truncated matrix (rows/columns $n = 0, 1, 2, 3, 4$):

$$\hat{H} \Rightarrow \begin{pmatrix} 0 & -E_J/2 & 0 & 0 & 0 \\ -E_J/2 & 4E_c & -E_J/2 & 0 & 0 \\ 0 & -E_J/2 & 16E_c & -E_J/2 & 0 \\ 0 & 0 & -E_J/2 & 36E_c & -E_J/2 \\ 0 & 0 & 0 & -E_J/2 & 64E_c \end{pmatrix}$$

This is a tridiagonal infinite matrix: diagonal $4E_c n^2$, sub-/super-diagonal $-E_J/2$.

(c) Restriction to $n = 0, 1$ only.

$$\hat{H}|_{n=0,1} = \begin{pmatrix} 0 & -E_J/2 \\ -E_J/2 & 4E_c \end{pmatrix}$$

Q.6 — Short answers

Solution:

(a) Hamiltonian of a superconducting (charge) qubit:

$$\hat{H} = 4E_c (\hat{n} - n_g)^2 - E_J \cos \hat{\phi}$$

(b) Transmon Hamiltonian (expanded around the bottom of the cosine well, $\hat{\phi} = (E_c/8E_J)^{1/4}(\hat{b} + \hat{b}^\dagger)$, $E_J \gg E_c$):

$$\hat{H} \approx \sqrt{8E_J E_c} \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) - \frac{E_c}{12} (\hat{b} + \hat{b}^\dagger)^4$$

(c) JC Hamiltonian under RWA (sign of qubit term chosen so that $\hat{\sigma}_z |g\rangle = +|g\rangle$ puts the ground at the lower energy):

$$\hat{H}_{JC} = \hbar\omega_r \hat{a}^\dagger \hat{a} - \frac{\hbar\omega_q}{2} \hat{\sigma}_z + \hbar g (\hat{a} \hat{\sigma}^\dagger + \hat{a}^\dagger \hat{\sigma})$$

i.e. $E_g = -\hbar\omega_q/2$, $E_e = +\hbar\omega_q/2$.

(d) Energy eigenstates and eigenvalues of JC under RWA (general detuning $\Delta = \omega_q - \omega_r$):

$$|n, +\rangle = \sin \theta_n |g, n+1\rangle + \cos \theta_n |e, n\rangle, \quad |n, -\rangle = \cos \theta_n |g, n+1\rangle - \sin \theta_n |e, n\rangle,$$

$$\tan(2\theta_n) = \frac{2g\sqrt{n+1}}{\Delta},$$

$$E_{n,\pm} = \hbar\omega_r \left(n + \frac{1}{2} \right) \pm \frac{\hbar}{2} \sqrt{\Delta^2 + 4g^2(n+1)}$$

Ground state: $|0, g\rangle$ with $E_{0,g} = -\hbar\omega_q/2$ (consistent with the convention sign of the qubit term in Q.6(c)).

(e) Coupled system in the dispersive regime ($|\Delta| \gg g\sqrt{n+1}$, $\chi = g^2/\Delta$):

$$\hat{H}_{\text{disp}} = \hbar(\omega_r + \chi \hat{\sigma}_z) \hat{a}^\dagger \hat{a} + \frac{\hbar\omega'_q}{2} \hat{\sigma}_z$$

with the Lamb-shifted qubit frequency $\omega'_q = \omega_q + \chi$.

(f) Rabi oscillations? N, however:

JC: Y — the term $\hbar g(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma})$ exchanges energy and produces vacuum/ n -photon Rabi oscillations.

Dispersive: N (no exchange Rabi, since the coupling is now purely diagonal in $\hat{a}^\dagger \hat{a}$ and $\hat{\sigma}_z$). However the qubit is still *drivable*: applying a classical microwave tone at ω'_q induces ordinary driven Rabi oscillations on the qubit itself.

(g) Hamiltonian of LC (linear) circuit in ladder operators:

$$\hat{H}_{LC} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \omega = \frac{1}{\sqrt{LC}}$$

(h) Matrix form of (g) in the Fock basis (diagonal):

$$\hat{H}_{LC} \Rightarrow \begin{pmatrix} \frac{\hbar\omega}{2} & 0 & 0 & 0 & \dots \\ 0 & \frac{3\hbar\omega}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{5\hbar\omega}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{7\hbar\omega}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(i) Dispersive qubit readout.

The cavity (effective) frequency depends on the qubit state through the $\chi \hat{\sigma}_z \hat{a}^\dagger \hat{a}$ term:

$$\omega_r \longrightarrow \omega_r \pm \chi \quad \text{for } |g\rangle/|e\rangle.$$

Probing the resonator transmission/reflection at a fixed frequency near ω_r therefore produces a different complex amplitude (phase & magnitude of S_{21}) for the two qubit states; demodulating the readout pulse yields the qubit state in a single shot.

(j) Dispersive photon-number measurement (number splitting).

The same dispersive coupling can be rewritten as a photon-number-dependent qubit frequency:

$$\omega'_q \longrightarrow \omega'_q + 2\chi \hat{n}_a.$$

A qubit spectroscopy then shows a comb of peaks at $\omega'_q + 2\chi n$, one per Fock state $|n\rangle$ in the cavity. The relative heights of the peaks give the photon-number distribution $P(n)$.

(k) Effect of detuning Δ on the Rabi frequency? ☐ — generalised Rabi $\Omega = \sqrt{\Omega_0^2 + \Delta^2}$.

(l) Does the Rabi frequency depend on the cavity Fock state? ☐ — the vacuum/ n -photon Rabi frequency is $2g\sqrt{n+1}$ in the JC regime. (In the dispersive regime there is no exchange Rabi, so this n -dependence disappears: **N** there.)

(m) How to verify qubit non-linearity? ☐ II: two-tone spectroscopy — using one tone to keep the resonator at ω_r for readout while sweeping a second tone across the qubit resolves the $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |2\rangle$ transitions, whose splitting $-E_c$ is the anharmonicity.

(n) Gate fidelity and Quantum Process Tomography.

Gate fidelity: a scalar measure of how close an implemented unitary/CP-map $\mathcal{E}$ is to the ideal U_{ideal} . For pure states $F = |\langle \psi_{\text{ideal}} | \psi_{\text{actual}} \rangle|^2$; for general channels, the average gate fidelity is $\bar{F} = \int d\psi \langle \psi | U^\dagger \mathcal{E}(|\psi\rangle\langle\psi|) U | \psi \rangle$.

Quantum Process Tomography (QPT): an experimental procedure that fully reconstructs the process matrix χ describing $\mathcal{E}$ by preparing a complete set of input states $\{\rho_i\}$, applying the gate, and performing state tomography on each output. Inverting the resulting linear system gives χ , from which any figure of merit (including the gate fidelity) can be computed.