

PHYS 514 — Final Exam Study Guide

Midterm material + final exam additions (measurements, JPA, VQE)

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Scope. Sections 1–15 cover everything from the midterm syllabus (still examinable). Sections 16–18 cover the new final-exam topics: quantum circuit measurements (cavity & qubit spectroscopy, Rabi, Ramsey, Bloch-sphere pulses), Josephson parametric amplifier, and the Variational Quantum Eigensolver (VQE). Excludes (per instructor): Set-1 Q5/Q6, Set-2 Q5/Q10, sideband modulation, measurements in the rotating frame, transmon fabrication details.

1. The Mechanical Harmonic Oscillator

Classical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad \omega = \sqrt{k/m}.$$

Hamilton’s equations reproduce $\ddot{x} + \omega^2 x = 0$.

Quantum Hamiltonian

Promote x, p to operators with $[\hat{x}, \hat{p}] = i\hbar$:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2.$$

Ladder (creation/annihilation) operators

Define

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right).$$

Inverse:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a}).$$

Commutators & Hamiltonian

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

Substitute into $\hat{H}$:

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}), \quad \hat{N} = \hat{a}^\dagger\hat{a}.$$

Eigenstates & eigenvalues

$\hat{N}|n\rangle = n|n\rangle$ with $n = 0, 1, 2, \dots$ and

$$E_n = \hbar\omega(n + \frac{1}{2}).$$

Ladder action:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|0\rangle = 0, \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

2. Lagrangian Formalism for LC Circuits

Node flux plays the role of position: $\phi(t) = \int_{-\infty}^t V(t') dt'$, so $V = \dot{\phi}$ and charge $Q = C\dot{\phi}$.

Energies in circuit elements

- Capacitor (kinetic-like): $T = \frac{1}{2}C\dot{\phi}^2$.
- Inductor (potential-like): $U = \frac{\phi^2}{2L}$ (from $U = \frac{1}{2}LI^2$ with $I = \phi/L$).

Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}C\dot{\phi}^2 - \frac{\phi^2}{2L}.$$

Euler–Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow C\ddot{\phi} + \phi/L = 0 \Rightarrow \omega_0 = 1/\sqrt{LC}.$$

Conjugate momentum & Hamiltonian

$$Q \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C\dot{\phi} \quad (\text{charge}), \quad H = Q\dot{\phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\phi^2}{2L}.$$

Quantization: $[\hat{\phi}, \hat{Q}] = i\hbar$ — direct analog of $[\hat{x}, \hat{p}] = i\hbar$.

3. Circuit Analysis — Multiple Inductors & Capacitors

Recipe for an arbitrary LC network:

1. Pick a **ground** node; assign node fluxes ϕ_i to the remaining nodes.
2. Each capacitor C_{ij} between nodes i, j contributes $\frac{1}{2}C_{ij}(\dot{\phi}_i - \dot{\phi}_j)^2$ to T .
3. Each inductor L_{ij} contributes $(\phi_i - \phi_j)^2/(2L_{ij})$ to U .
4. $\mathcal{L} = T - U$; Euler–Lagrange gives coupled equations of motion.
5. Conjugate charges $Q_i = \partial \mathcal{L} / \partial \dot{\phi}_i$; Legendre-transform to $H(Q_i, \phi_i)$.

Example — two LC nodes coupled by a capacitor C_g .

$$\mathcal{L} = \frac{1}{2}C_1\dot{\phi}_1^2 + \frac{1}{2}C_2\dot{\phi}_2^2 + \frac{1}{2}C_g(\dot{\phi}_1 - \dot{\phi}_2)^2 - \frac{\phi_1^2}{2L_1} - \frac{\phi_2^2}{2L_2}.$$

The capacitance matrix $\mathbf{C}$ is positive-definite; $H = \frac{1}{2}\vec{Q}^T \mathbf{C}^{-1} \vec{Q} + \dots$, with mode frequencies obtained by simultaneously diagonalising $\mathbf{C}^{-1}$ and $\mathbf{L}^{-1}$.

4. The LC Oscillator — Classical $\rightarrow$ Quantum

The Hamiltonian $H = Q^2/2C + \phi^2/2L$ is a harmonic oscillator with mapping

$$x \rightarrow \phi, \quad p \rightarrow Q, \quad m \rightarrow C, \quad \omega \rightarrow \omega_0 = 1/\sqrt{LC}.$$

Ladder representation

$$\hat{\phi} = \Phi_{\text{ZPF}} (\hat{a} + \hat{a}^\dagger), \quad \hat{Q} = -iQ_{\text{ZPF}} (\hat{a} - \hat{a}^\dagger),$$

with

$$\Phi_{\text{ZPF}} = \sqrt{\frac{\hbar}{2C\omega_0}} = \sqrt{\frac{\hbar Z}{2}}, \quad Q_{\text{ZPF}} = \sqrt{\frac{\hbar C\omega_0}{2}} = \sqrt{\frac{\hbar}{2Z}}, \quad Z = \sqrt{L/C}.$$

Hamiltonian & spectrum

$$\hat{H}_{LC} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right).$$

Wave functions in flux representation

With $\hat{a}\Psi_0 = 0$ and $\hat{a}^\dagger = \hat{\phi}/\Phi_{\text{ZPF}} - \hat{a}$,

$$\Psi_0(\phi) = \frac{1}{(2\pi\Phi_{\text{ZPF}}^2)^{1/4}} \exp\left(-\frac{\phi^2}{4\Phi_{\text{ZPF}}^2}\right),$$

$$\Psi_1(\phi) = \frac{\phi}{\Phi_{\text{ZPF}}} \Psi_0(\phi), \quad \Psi_2(\phi) = \left[(\phi/\Phi_{\text{ZPF}})^2 - 1 \right] \Psi_0(\phi).$$

Useful voltage ZPF

$$V_{\text{ZPF}} = Q_{\text{ZPF}}/C = \sqrt{\hbar\omega_0/(2C)}.$$

Worked numeric. $L = 50 \text{ nH}$, $C = 70 \text{ pF} \Rightarrow \omega_0 = 5.35 \times 10^8 \text{ rad/s}$ ($f_0 \approx 85 \text{ MHz}$), $Z = 26.7 \Omega$, $V_{\text{ZPF}} \approx 20 \text{ nV}$.

5. Number (Fock) States

Defining properties

- Orthonormal: $\langle m|n\rangle = \delta_{mn}$.
- Complete: $\sum_n |n\rangle\langle n| = I$.
- Eigenstates of $\hat{N}$: $\hat{N}|n\rangle = n|n\rangle$, and hence of $\hat{H}$.

Commutation relations involving $\hat{N}$

Derive from $[\hat{a}, \hat{a}^\dagger] = 1$:

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = 1.$$

Proof of the first: $[\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}] \hat{a} = -\hat{a}$. ■

Action of ladder operators on Fock states

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|0\rangle = 0.$$

Matrix elements: $\langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1}$, $\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{m,n+1}$. So $\hat{a}, \hat{a}^\dagger$ have **no** diagonal matrix elements — any expectation $\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^\dagger|n\rangle = 0$.

Fock-state superposition example

If $|\psi\rangle = \sqrt{0.2}|10\rangle + \sqrt{0.8}|40\rangle$, then $\bar{n} = \langle \hat{N} \rangle = 34$; after $\hat{a}$, the normalized state has $\bar{n} = 37.24$ — annihilation can *increase* the mean because it reweights amplitudes by $\sqrt{n}$.

6. Expectation Values, Uncertainty & Zero-Point Fluctuations

Fock-state expectations

For the LC oscillator in $|n\rangle$:

$$\langle \hat{\phi} \rangle = \langle \hat{Q} \rangle = 0, \quad \langle \hat{\phi}^2 \rangle = \Phi_{\text{ZPF}}^2(2n+1), \quad \langle \hat{Q}^2 \rangle = Q_{\text{ZPF}}^2(2n+1).$$

Proof uses $(\hat{a} + \hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{N} + 1$ and $\langle n|\hat{a}^2|n\rangle = 0$.

Uncertainty relation

$$\Delta\phi \Delta Q = \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}.$$

Vacuum $|0\rangle$ **saturates** Heisenberg: $\Delta\phi \Delta Q = \hbar/2$.

Zero-Point Fluctuations (ZPF)

Even in the ground state the oscillator is not at rest:

$$\langle \hat{\phi}^2 \rangle_0 = \Phi_{\text{ZPF}}^2, \quad \langle \hat{Q}^2 \rangle_0 = Q_{\text{ZPF}}^2, \quad \langle \hat{V}^2 \rangle_0 = V_{\text{ZPF}}^2.$$

Physical consequence: measurable quantum noise floor on voltage/flux/field even in absence of photons.

Quadrature operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger).$$

- $[\hat{X}_1, \hat{X}_2] = i/2 \Rightarrow \Delta X_1 \Delta X_2 \geq 1/4$.
 - Vacuum: $\Delta X_i^2 = 1/4$.
 - Fock $|n\rangle$: $\Delta X_i^2 = n/2 + 1/4$.
 - Coherent $|\alpha\rangle$: $\Delta X_i^2 = 1/4$ (minimum-uncertainty).
-

7. Quantum Dynamics

Schrödinger picture

States evolve, operators are static:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle, \quad i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle.$$

For the QHO with $|\psi(0)\rangle = \sum_n c_n |n\rangle$:

$$|\psi(t)\rangle = \sum_n c_n e^{-i(n+1/2)\omega t} |n\rangle.$$

Each Fock amplitude phases at a **different rate** $n\omega$; the $\frac{1}{2}$ is a global phase (drops out of every expectation value / probability).

Heisenberg picture

Operators evolve:

$$\hat{O}_H(t) = e^{i\hat{H}t/\hbar} \hat{O} e^{-i\hat{H}t/\hbar}, \quad \frac{d\hat{O}_H}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{O}_H].$$

For QHO: $\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$, $\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{+i\omega t}$. Hence $\hat{x}(t) = \hat{x}(0) \cos \omega t + \frac{\hat{p}(0)}{m\omega} \sin \omega t$ — the operator form of classical motion.

Equivalence

Expectation values match:

$$\langle \psi(t) | \hat{O} | \psi(t) \rangle_S = \langle \psi(0) | \hat{O}_H(t) | \psi(0) \rangle_H.$$

8. Field Quantization

Classical EM in a cavity

For a 1-D cavity of length L , the mode with wave-number $k = n\pi/L$ has

$$E(z, t) = \mathcal{E}_0 \sin(kz) q(t), \quad B(z, t) = \frac{\mathcal{E}_0}{c} \cos(kz) \dot{q}(t)/\omega,$$

where $q(t)$ obeys the HO equation of motion.

Quantized fields

Promote $q \rightarrow \hat{q}$, introduce ladder operators as in §1. Then

$$\hat{E}(z) = \mathcal{E}_0 \sin(kz) (\hat{a} + \hat{a}^\dagger), \quad \hat{B}(z) = i\mathcal{B}_0 \cos(kz) (\hat{a}^\dagger - \hat{a})$$

with $\mathcal{E}_0 = \sqrt{\hbar\omega/(\varepsilon_0 V)}$, $\mathcal{B}_0 = \mathcal{E}_0/c$.

EM-wave Hamiltonian in ladder form

$$\hat{H}_{\text{EM}} = \frac{1}{2} \int (\varepsilon_0 \hat{E}^2 + \mu_0^{-1} \hat{B}^2) dV = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}).$$

Each cavity mode is an independent QHO; states $|n\rangle$ describe n **photons** of that mode.

9. Commutation Relations — Field & Fock Operators

Ladder operators

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger, \quad [\hat{N}, \hat{a}] = -\hat{a}.$$

Electric and magnetic field operators

Using $\hat{E} \propto (\hat{a} + \hat{a}^\dagger)$, $\hat{B} \propto i(\hat{a}^\dagger - \hat{a})$:

$$[\hat{E}(z), \hat{B}(z')] = i\mathcal{E}_0\mathcal{B}_0 \sin(kz) \cos(kz') \underbrace{[(\hat{a} + \hat{a}^\dagger), (\hat{a}^\dagger - \hat{a})]}_{=2[\hat{a}, \hat{a}^\dagger]=2},$$

$$[\hat{E}(z), \hat{B}(z')] = 2i\mathcal{E}_0\mathcal{B}_0 \sin(kz) \cos(kz') \neq 0.$$

E and B are **non-commuting** observables of the electromagnetic mode — one cannot measure both simultaneously with arbitrary precision (field-component uncertainty relation).

Fock matrix elements of the fields

$\langle n|\hat{E}|n\rangle = 0$, $\langle n|\hat{B}|n\rangle = 0$; $\langle n|\hat{E}^2|n\rangle = \mathcal{E}_0^2 \sin^2(kz)(2n+1)$, etc. $\Rightarrow \Delta E = \mathcal{E}_0 |\sin(kz)| \sqrt{2n+1}$; for $|7\rangle$, $\Delta E = \mathcal{E}_0 |\sin(kz)| \sqrt{15}$.

10. Coherent States

Definition & construction

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\alpha \in \mathbb{C}$. Displacement construction: $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ with $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$. Fock-state expansion (derived by expanding $|\alpha\rangle = \sum c_n|n\rangle$ in $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and using $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$):

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

Proof of normalization

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2} \sum_{m,n} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \langle m|n\rangle = e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2} e^{|\alpha|^2} = 1. \blacksquare$$

Photon-number (Fock) distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \quad (\text{Poisson with mean } \bar{n} = |\alpha|^2).$$

Expectation values

$$\langle\hat{N}\rangle = |\alpha|^2, \quad \Delta N = |\alpha|, \quad \langle\hat{a}\rangle = \alpha, \quad \langle\hat{a}^\dagger\rangle = \alpha^*, \quad \langle\hat{a}^\dagger\hat{a}\rangle = |\alpha|^2.$$

For $\alpha = |\alpha|e^{i\theta}$: $\langle X_1\rangle = |\alpha| \cos \theta$, $\langle X_2\rangle = |\alpha| \sin \theta$, $\Delta X_i^2 = 1/4$.

Why $\hat{a}$ — not $\hat{a}^\dagger$ — has eigenstates

$\hat{a}$ is non-Hermitian, so spectrum is complex and eigenstates (coherent states) are non-orthogonal / over-complete. Trying $\hat{a}^\dagger|\psi\rangle = \lambda|\psi\rangle$ forces $c_n = c_0\sqrt{n!}/\lambda^n$ which is **not normalizable** — so $\hat{a}^\dagger$ has *no* eigenstates.

11. Superconductivity Fundamentals

Property	Description
Zero DC resistance	Persistent supercurrents below T_c ; $R \rightarrow 0$.
Two-fluid behavior	Coexisting normal (n_n) and superfluid (n_s) densities; $n_s \rightarrow n$ as $T \rightarrow 0$.
Energy gap 2Δ	Cooper-pair binding; BCS: $2\Delta \approx 3.52 k_B T_c$.
Meissner effect	B expelled from bulk; decay length λ_L (perfect diamagnet).
Long-range phase coherence	Macroscopic order parameter $\psi = \psi e^{i\varphi(\vec{r})}$; basis for flux quantization and Josephson physics.

Cooper pairs. Effective charge $2e$, formed by phonon-mediated attraction; boson-like condensate at $T < T_c$.

12. Quantization of Physical Quantities (conceptual)

Quantity	Quantum	Value	Origin
Charge	Cooper-pair charge	$2e$	Cooper pairs are the charge carriers in a superconductor.
Magnetic flux	Flux quantum Φ_0	$h/2e \approx 2.07 \times 10^{-15}$ Wb	Single-valuedness of the macroscopic wave function around a loop.
Resistance	von Klitzing constant R_K	$h/e^2 \approx 25.813 \text{ k}\Omega$	Quantum Hall effect; appears in Josephson/transmon as $R_Q = h/(2e)^2$.

No derivations required — just remember the units, numerical values, and the physical meaning (flux quantization is what makes SQUIDS and flux qubits possible).

13. The Josephson Junction

Two Josephson equations

$$I = I_c \sin \varphi, \quad \frac{d\varphi}{dt} = \frac{2eV}{\hbar} = \frac{2\pi V}{\Phi_0},$$

where φ is the gauge-invariant phase difference of the superconducting order parameter across the barrier.

DC Josephson effect

$V = 0$: a DC supercurrent $I \in [-I_c, I_c]$ flows at constant φ with **no** voltage drop.

AC Josephson effect

Constant $V \neq 0 \Rightarrow \varphi(t) = \omega_J t$ with $\omega_J = 2eV/\hbar$. Current oscillates at

$$f_J = V/\Phi_0 \approx 483.6 \text{ MHz}/\mu\text{V}.$$

Josephson energy E_J

From $P = IV = dE/dt$ and the two JJ equations,

$$\frac{dE}{dt} = I_c \sin \varphi \cdot \frac{\hbar}{2e} \dot{\varphi} \Rightarrow E(\varphi) = -E_J \cos \varphi, \quad E_J = \frac{\hbar I_c}{2e} = \frac{I_c \Phi_0}{2\pi}.$$

In flux variables: $E = -E_J \cos(2\pi\Phi/\Phi_0)$.

Josephson inductance

$V = L_J \dot{I}$ combined with the JJ equations gives

$$L_J = \frac{\Phi_0}{2\pi I_c \cos \varphi} = \frac{L_{J0}}{\cos \varphi} \quad (\text{non-linear; depends on } \varphi).$$

LC parabola vs JJ cosine

Feature	LC inductor	Josephson junction
Potential	$\phi^2/2L$ (parabola)	$-E_J \cos(2\pi\Phi/\Phi_0)$ (cosine)
Spectrum	Equal spacing $\hbar\omega_0$	Unequal (anharmonic)
Inductance	Linear, constant	$L_J \propto 1/\cos \varphi$, diverges at $\varphi = \pi/2$

The anharmonicity of the cosine is what lets us pick out a two-level subspace for qubits.

14. Non-Linear Oscillator — LC + Josephson Junction

Hamiltonian

A JJ shunted by capacitance C (ignore any parallel linear inductor for the pure transmon):

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos \hat{\varphi}, \quad E_C = \frac{e^2}{2C}, \quad [\hat{\varphi}, \hat{n}] = i.$$

Plasma (small-oscillation) frequency

Expand $\cos \hat{\varphi} \approx 1 - \hat{\varphi}^2/2 + \hat{\varphi}^4/24 - \dots$:

$$\hat{H} \approx \frac{\hat{Q}^2}{2C} + \frac{E_J}{2}\hat{\varphi}^2 + \underbrace{-\frac{E_J}{24}\hat{\varphi}^4}_{\text{non-linear}}.$$

Harmonic part gives plasma frequency

$$\omega_p = \sqrt{8E_J E_C}/\hbar.$$

Origin & implications of non-linearity

- The $\hat{\varphi}^4$ term couples Fock levels unevenly $\rightarrow$ **anharmonicity** $\alpha \approx -E_C/\hbar$.
- Level spacings $E_{n+1} - E_n \simeq \hbar\omega_p + \alpha n$ — decrease with n .
- Consequence: $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |2\rangle$ transitions have *different* frequencies, so resonant pulses at ω_{01} drive only the qubit subspace (provided pulse bandwidth $\ll |\alpha|$).
- Without non-linearity (pure LC) all transitions are degenerate — a harmonic oscillator can never be used as a qubit.

Transmon regime

$E_J/E_C \gtrsim 50 \rightarrow$ charge-dispersion bands flatten (exponentially small in $\sqrt{8E_J/E_C} \rightarrow$ charge-noise insensitive. Trade-off: smaller anharmonicity ($\sim -E_C$, a few hundred MHz), compensated by pulse shaping (e.g. DRAG).

Three basic JJ qubits

- **Charge (CPB):** $E_C \gg E_J$; computational states are charge eigenstates; charge-noise sensitive.
- **Flux:** JJ loop with $\Phi \approx \Phi_0/2$; double-well; persistent-current basis.
- **Phase:** current-biased JJ; tilted-washboard; two lowest bound states in a metastable well.

Transmon = charge qubit pushed to $E_J \gg E_C$.

15. Qubit–Cavity Interaction (Jaynes–Cummings)

Full Hamiltonian

$$\hat{H} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2} \hbar\omega_q \hat{\sigma}_z + \hbar g (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-).$$

- ω_r : cavity frequency; ω_q : qubit frequency; g : qubit–cavity coupling.
- Interaction term (RWA): exchanges one photon with one qubit excitation — conserves total excitation number $\hat{N}_{\text{exc}} = \hat{a}^\dagger \hat{a} + |e\rangle\langle e|$.

Action on bare states

$$\hat{H}_{\text{int}}|n, 0\rangle = \hbar g \sqrt{n} |n-1, 1\rangle, \quad \hat{H}_{\text{int}}|n, 1\rangle = \hbar g \sqrt{n+1} |n+1, 0\rangle.$$

Notation: $|n, 0\rangle = |n\rangle_{\text{cav}} \otimes |g\rangle$, $|n, 1\rangle = |n\rangle_{\text{cav}} \otimes |e\rangle$.

Dressed eigenstates (excitation- $n+1$ subspace)

Diagonalize in $\{|g, n+1\rangle, |e, n\rangle\}$:

$$|n, +\rangle = \sin \theta_n |g, n+1\rangle + \cos \theta_n |e, n\rangle, \quad |n, -\rangle = \cos \theta_n |g, n+1\rangle - \sin \theta_n |e, n\rangle,$$

with $\tan 2\theta_n = \frac{2g\sqrt{n+1}}{\Delta}$, $\Delta = \omega_q - \omega_r$.

On resonance ($\Delta = 0$)

$\theta_n = \pi/4$:

$$|n, \pm\rangle = \frac{1}{\sqrt{2}}(|g, n+1\rangle \pm |e, n\rangle),$$

$$E_{n,\pm} = \hbar\omega(n + \frac{1}{2}) \pm \hbar g \sqrt{n+1}.$$

Vacuum-Rabi splitting $2\hbar g \sqrt{n+1}$; for $n = 0$ it is $2\hbar g$.

Dispersive limit ($|\Delta| \gg g\sqrt{n+1}$)

$\theta_n \rightarrow 0$: $|n, -\rangle \rightarrow |g, n+1\rangle$, $|n, +\rangle \rightarrow |e, n\rangle$. Effective Hamiltonian:

$$\hat{H}_{\text{eff}} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar(\omega_q + 2\chi \hat{a}^\dagger \hat{a}) \hat{\sigma}_z, \quad \chi = g^2/\Delta.$$

Qubit shifts cavity frequency by χ per photon $\rightarrow$ basis of **QND dispersive readout**.

Time evolution from $|\psi(0)\rangle = |0, 1\rangle$ (vacuum-Rabi oscillations)

Using $\hat{H}_{\text{int}}^2|0, 1\rangle = (\hbar g)^2|0, 1\rangle$,

$$|\psi(t)\rangle = \cos(gt)|0, 1\rangle - i\sin(gt)|1, 0\rangle.$$

Populations:

$$P_{|0,1\rangle}(t) = \cos^2(gt), \quad P_{|1,0\rangle}(t) = \sin^2(gt).$$

Probabilities & expectation values — examples

- **Complete transfer qubit $\rightarrow$ cavity:** $t_{\text{swap}} = \pi/(2g)$.
- **Entangling half-transfer** ($t = \pi/(4g)$): $|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle - i|1, 0\rangle)$ — maximally entangled cavity-qubit Bell-like state.
- **Cavity photon-number expectation** in the evolving state: $\langle \hat{a}^\dagger \hat{a} \rangle(t) = \sin^2(gt)$.
- **Qubit inversion:** $\langle \hat{\sigma}_z \rangle(t) = 1 - 2\sin^2(gt) = \cos(2gt)$ — oscillates at the vacuum-Rabi frequency $2g$.

Direct qubit–qubit exchange coupling

Replace cavity mode by a second qubit:

$$\hat{H} = \frac{1}{2}\hbar\omega_1 \hat{\sigma}_z^{(1)} + \frac{1}{2}\hbar\omega_2 \hat{\sigma}_z^{(2)} + \hbar J(\hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)}).$$

Quick Formula Sheet

Quantity	Formula
QHO eigenvalue	$E_n = \hbar\omega(n + 1/2)$
Ladder action	$\hat{a} n\rangle = \sqrt{n} n-1\rangle$, $\hat{a}^\dagger n\rangle = \sqrt{n+1} n+1\rangle$
Commutators	$[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$, $[\hat{\phi}, \hat{Q}] = i\hbar$
LC frequency	$\omega_0 = 1/\sqrt{LC}$, $Z = \sqrt{L/C}$
ZPF	$\Phi_{\text{ZPF}} = \sqrt{\hbar Z/2}$, $Q_{\text{ZPF}} = \sqrt{\hbar/2Z}$, $V_{\text{ZPF}} = \sqrt{\hbar\omega_0/2C}$
Fock uncertainty	$\Delta\phi\Delta Q = \hbar(2n+1)/2$
Quadrature commutator	$[\hat{X}_1, \hat{X}_2] = i/2$
Field operators	$\hat{E} \propto (\hat{a} + \hat{a}^\dagger)$, $\hat{B} \propto i(\hat{a}^\dagger - \hat{a})$, $[\hat{E}, \hat{B}] \neq 0$
Flux quantum	$\Phi_0 = h/2e \approx 2.07 \times 10^{-15}$ Wb
Resistance quantum	$R_K = h/e^2 \approx 25.8$ k Ω

Quantity	Formula
Josephson equations	$I = I_c \sin \varphi, \dot{\varphi} = 2eV/\hbar$
Josephson energy	$E_J = \hbar I_c / (2e)$
Josephson inductance	$L_J = \Phi_0 / (2\pi I_c \cos \varphi)$
Transmon ω_p	$\omega_p = \sqrt{8E_J E_C} / \hbar, E_C = e^2 / 2C$
JC interaction	$\hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-)$
Vacuum-Rabi osc	$\ \psi(t)\rangle = \cos(gt)\ 0, 1\rangle - i \sin(gt)\ 1, 0\rangle$
Vacuum-Rabi energies	$E_{n,\pm} = \hbar\omega(n + \frac{1}{2}) \pm \hbar g\sqrt{n+1}$
Dispersive shift	$\chi = g^2 / \Delta$
Coherent state	$\hat{a}\ \alpha\rangle = \alpha\ \alpha\rangle, P(n) = e^{-\ \alpha\ ^2} \ \alpha\ ^{2n} / n!$
Rabi (driven qubit)	$P_e(t) = \frac{A^2}{\Omega_R^2} \sin^2(\Omega_R t / 2), \Omega_R = \sqrt{A^2 + \Delta_d^2}$
$\pi / \pi/2$ pulse area	$\Omega_R t_\pi = \pi, \Omega_R t_{\pi/2} = \pi/2$
Ramsey fringe	$P_e(t) = A + B \sin(2\pi\Delta_d t + \phi) \exp(-t/T_2^*)$
Dispersive readout shift	cavity peak shifts to $\omega_r \pm \chi$ for qubit in $\ 0\rangle/\ 1\rangle$
Quality factor	$1/Q_{\text{tot}} = 1/Q_{\text{int}} + 1/Q_{\text{ext}}$
JPA (3-wave) gain	$G \sim 1/(1 - (\xi/\xi_c)^2)$ near threshold
VQE estimator	$E(\vec{\theta}) = \langle \psi(\vec{\theta}) \hat{H} \psi(\vec{\theta}) \rangle \geq E_0$

16. Quantum Circuit Measurements

The standard cQED characterization workflow has **five experiments** in two groups:

Domain	Experiment	What it measures
Frequency	One-tone “punch-out”	Bare/dressed cavity frequency ω_r ; confirms qubit is “alive”
Frequency	Two-tone spectroscopy	Qubit frequency $\omega_q = \omega_{ge}$ (and two-photon $\omega_{gf}/2$)
Time	Rabi	Drive amplitude A , calibrate $\pi / \pi/2$ pulse durations
Time	T_1	Energy-relaxation lifetime
Time	Ramsey	Pure-dephasing time T_2^* and fine ω_q correction

Frequency-domain measurements look only at scattering parameters (S_{21} transmitted, S_{11} reflected) via a Vector Network Analyzer (VNA). Time-domain measurements require an **initialization** $\rightarrow$ **preparation/manipulation** $\rightarrow$ **readout** sequence using gated I/Q-mixer pulses from an AWG plus homodyne detection.

16.1 Cavity characterization (linewidth, Q)

Measure S_{21} vs frequency; the cavity transmission is a Lorentzian

$$|S_{21}(\omega)|^2 = \frac{A}{(\omega - \omega_r)^2 + (\kappa/2)^2}.$$

- **Resonance frequency** ω_r = peak (max transmission) / dip (min reflection).
- **Linewidth** $\kappa/2\pi$ = full width at half maximum ($\equiv$ 3 dB drop in log scale).
- **Phase** sweeps by π across resonance with slope $d\theta/d\omega \approx 2/\kappa$.
- **Quality factor decomposition:**

$$\frac{1}{Q_{\text{tot}}} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}}, \quad Q_{\text{tot}} = \omega_r/\kappa.$$

16.2 One-tone spectroscopy — the “punch-out”

Goal: verify a qubit is coupled to the cavity (without yet knowing ω_q).

Procedure: measure S_{21} of the cavity at **low drive power** and again at **high drive power**.

Power	Photon number	What you see
Low (~ -60 dBm)	$\bar{n} \ll 1$	Qubit in $ g\rangle \Rightarrow$ dressed cavity at $\omega_r^{\text{dressed}} = \omega_r - \chi$
High ($\sim +10$ dBm)	Drives above I_c	Qubit “washed out” $\Rightarrow$ bare cavity at ω_r^{bare}

A frequency shift between the two traces = “the cavity has punched out” = a working qubit is present. No shift $\Rightarrow$ junction is dead.

16.3 Two-tone spectroscopy — finding ω_q

Setup: VNA continuously probes the cavity at $\omega_r^{\text{dressed}}$ (constant high $|S_{21}|$). A second tone (BNC) is swept in frequency.

- When BNC frequency $\neq \omega_q$: qubit stays in $|g\rangle$, transmission stays high.
- When **BNC** = ω_{ge} : drive excites the qubit, dispersive shift moves the cavity off-resonant with the VNA tone, and $|S_{21}|$ **dips**.
- At higher BNC power, a second dip appears at $(\omega_{ge} + \omega_{gf})/2$ — the **two-photon** $|g\rangle \rightarrow |f\rangle$ transition. Anharmonicity:

$$\alpha = \omega_{ef} - \omega_{ge} \approx -E_C.$$

16.4 Time-domain building blocks

I/Q mixer as a switch. A microwave LO at frequency ω_q is gated by DC pulses on the I and Q ports:

$$\text{RF out} = I(t) \cos(\omega_q t) + Q(t) \cos(\omega_q t + \pi/2).$$

- Pulse on **I** $\Rightarrow$ in-phase $\Rightarrow$ rotation about the x -axis on the Bloch sphere. - Pulse on **Q** $\Rightarrow$ 90°-out-of-phase $\Rightarrow$ rotation about y -axis.

Bloch-sphere pulse table (qubit starts in $|g\rangle$ on $+z$):

Pulse	Duration	Rotation axis	Final state	Bloch vector
π on I	$t_\pi = \pi/\Omega_R$	x by π	$ e\rangle$	$-\hat{z}$
$\pi/2$ on I	$t_{\pi/2} = \pi/(2\Omega_R)$	x by $\pi/2$	$\frac{1}{\sqrt{2}}(g\rangle - i e\rangle)$	$-\hat{y}$

Pulse	Duration	Rotation axis	Final state	Bloch vector
$\pi/2$ on Q	$t_{\pi/2}$	y by $\pi/2$	$\frac{1}{\sqrt{2}}(g\rangle + e\rangle)$	$+\hat{x}$
π on Q	t_{π}	y by π	$ e\rangle$	$-\hat{z}$

For arbitrary target Bloch angles (θ, ϕ) : pulse area $\Omega_R t = \theta$, drive phase $= \phi$.

Homodyne readout. Send a coherent tone at ω_c to the cavity, split off a local-oscillator copy, then mix the reflected signal back with the LO:

$$I \propto \cos \phi_q, \quad Q \propto \sin \phi_q \approx \phi_q,$$

where $\phi_q = \pm 2\chi/\kappa$ depending on qubit state. Sign of $Q \rightarrow$ qubit state. Repeat $N > 100$ shots:

$$P_g = \frac{N_+}{N_+ + N_-} \pm \sqrt{N_+ N_- / N^3}, \quad P_e = 1 - P_g.$$

16.5 Rabi measurement

Sequence: **drive pulse of duration** t at $\omega_q \rightarrow$ **readout pulse** at ω_c . Sweep $t \in [0, 100 \text{ ns}]$, average $N \sim 200$ shots per step.

Excited-state population (from solving the driven 2-level Schrödinger equation with drive amplitude $A = Ed$, detuning $\Delta_d = \omega_q - \omega_d$):

$$P_e(t) = \frac{A^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R t}{2}\right), \quad \Omega_R = \sqrt{A^2 + \Delta_d^2}.$$

Drive detuning	Effect
$\Delta_d = 0$ (on-resonance)	Maximum contrast (oscillates between 0 and 1), slowest Rabi frequency $\Omega_R = A$
$\Delta_d \neq 0$	Lower contrast $(A/\Omega_R)^2 < 1$, faster Ω_R

Plotting $P_e(t, \omega_d)$ as a 2-D map gives the **chevron pattern** — the minimum- Ω_R point on the chevron pinpoints ω_q exactly. From the on-resonant fit you read off t_{π} and $t_{\pi/2}$ used for all subsequent state preparation.

16.6 T_1 (relaxation) measurement

1. π -pulse $\rightarrow$ qubit in $|e\rangle$.
2. Wait time t (no drive).
3. Readout.

Fit $P_e(t) = A + B e^{-t/T_1}$. Typical transmon T_1 : 10–100 μs .

16.7 Ramsey measurement (T_2^*)

Sequence: $\pi/2$ pulse $\rightarrow$ wait $t \rightarrow$ second $\pi/2$ pulse $\rightarrow$ readout.

Physically: first $\pi/2$ puts the qubit on the equator (along $+\hat{x}$, say). During the wait, the Bloch vector precesses at $\Delta_d = \omega_q - \omega_d$ in the rotating frame. The second $\pi/2$ projects the accumulated phase back onto $\hat{z}$.

Fit:

$$P_e(t) = A + B \sin(2\pi\Delta_d t + \phi) e^{-t/T_2^*}.$$

- **Oscillation frequency** = $\Delta_d \Rightarrow$ refines ω_q to sub-MHz precision.
- **Decay envelope** = T_2^* (pure-dephasing + relaxation contribution). For a T_1 -limited qubit $1/T_2^* = 1/(2T_1) + 1/T_\phi$.
- On-resonance ($\Delta_d = 0$): pure exponential decay, no oscillations (blue trace in Fig. 3.22).

16.8 Full state tomography

A single readout measures $\langle\sigma_z\rangle$. To reconstruct the full Bloch vector (x, y, z) , repeat the experiment $3\times$ with a tomographic pulse just before readout:

Pre-readout pulse	Measures
(none)	$z = \langle\sigma_z\rangle = P_g - P_e$
$\pi/2$ on Q (rotates $+x \rightarrow +z$)	$x = \langle\sigma_x\rangle$
$\pi/2$ on I (rotates $+y \rightarrow +z$)	$y = \langle\sigma_y\rangle$

17. Josephson Parametric Amplifier (JPA) — Basic Concept

Why amplification matters in cQED

The dispersive readout signal at the chip is at the **single-photon level** ($\sim 10^{-19}$ W). Cryogenic HEMT amplifiers add noise of ~ 10 photons; we need a **near-quantum-limited** first-stage amplifier mounted at the mixing-chamber stage. The JPA fills this role.

Core idea

A JPA is a nonlinear LC resonator whose inductance is provided by a Josephson junction (or SQUID, for tunability). The **nonlinear** Josephson inductance $L_J(\varphi) = L_{J0}/\cos\varphi$ makes the resonator's resonance frequency depend on its own oscillation amplitude (Kerr nonlinearity, term $-\frac{E_C}{12}(\hat{b} + \hat{b}^\dagger)^4$ from §14).

When a **strong pump** at ω_p drives the JPA, the nonlinearity mixes the pump with a weak signal at ω_s and produces gain through one of two parametric processes:

Mode	Pumping condition	Output
3-wave mixing	$\omega_p = \omega_s + \omega_i$	Signal + idler at $\omega_i = \omega_p - \omega_s$
4-wave mixing	$2\omega_p = \omega_s + \omega_i$	Degenerate when $\omega_s = \omega_i = \omega_p$

Two operating modes

- **Phase-preserving (non-degenerate):** signal and idler at different frequencies. Adds $\frac{1}{2}\hbar\omega$ of vacuum noise — **the standard quantum limit**.
- **Phase-sensitive (degenerate):** signal and idler coincide. Amplifies one quadrature, squeezes the other; can in principle add **zero** noise to the amplified quadrature (squeezing below the vacuum).

Tuning the JPA

A flux-tunable SQUID-based JPA has $E_J(\Phi_{\text{ext}}) = E_{J0}|\cos(\pi\Phi_{\text{ext}}/\Phi_0)|$, so an external flux line tunes the plasma frequency $\omega_p = \sqrt{8E_J(\Phi)E_C}/\hbar$ to match the signal frequency.

Performance figures of merit

- **Gain** G (typ. 20 dB) over a narrow band (~ 10 – 50 MHz).
- **Added noise** $\approx \frac{1}{2}$ photon (quantum-limited) vs. ~ 10 for a HEMT.
- **Bandwidth** $\times \sqrt{G}$ = constant (gain–bandwidth product) — fundamental limitation of single-pole JPAs (motivates JTWPA designs).
- **Saturation power** $\approx$ a few photons in the resonator (dynamic-range bottleneck of cQED).

(Detailed derivation of the parametric gain equation is **not** required — only the basic concept.)

18. Variational Quantum Eigensolver (VQE)

What problem does VQE solve?

Find the **ground-state energy** E_0 (and approximate ground state $|\psi_0\rangle$) of a Hamiltonian $\hat{H}$ — usually an electronic-structure Hamiltonian for a molecule, written in the qubit basis after a fermion-to-qubit map (Jordan–Wigner, Bravyi–Kitaev).

VQE is the leading **NISQ-era** algorithm because it (i) needs only short-depth circuits, (ii) is robust to many noise channels, and (iii) leverages a classical optimizer for the hard part.

The variational principle

For *any* normalized trial state $|\psi(\vec{\theta})\rangle$,

$$E(\vec{\theta}) \equiv \langle \psi(\vec{\theta}) | \hat{H} | \psi(\vec{\theta}) \rangle \geq E_0.$$

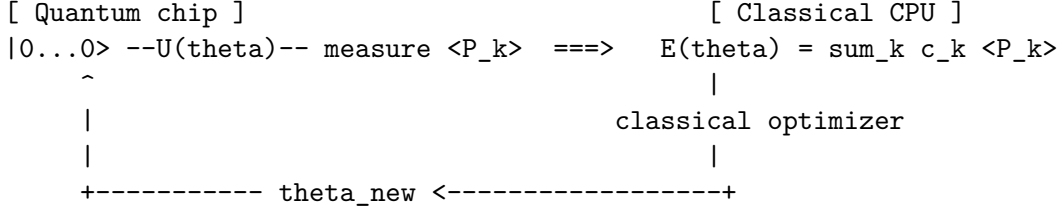
Equality only when $|\psi(\vec{\theta})\rangle = |\psi_0\rangle$. VQE searches the parameter space $\vec{\theta} \in \mathbb{R}^p$ to **minimize** $E(\vec{\theta})$.

The hybrid quantum–classical loop

1. **Encode** $\hat{H}$ as a sum of weighted Pauli strings:

$$\hat{H} = \sum_k c_k \hat{P}_k, \quad \hat{P}_k \in \{I, X, Y, Z\}^{\otimes n}.$$

2. **Prepare** $|\psi(\vec{\theta})\rangle = \hat{U}(\vec{\theta})|0\dots 0\rangle$ on the quantum processor using a parameterized **ansatz circuit** (UCCSD chemistry-inspired or hardware-efficient layers of R_y, R_z + entanglers).
3. **Measure** each Pauli term separately (rotate qubits to the Z basis if needed, sample N shots) to estimate $\langle \hat{P}_k \rangle$.
4. **Sum** classically: $E(\vec{\theta}) = \sum_k c_k \langle \hat{P}_k \rangle$.
5. **Update** $\vec{\theta}$ with a classical optimizer (gradient-free: COBYLA, Nelder–Mead; or parameter-shift gradients with SPSA / Adam).
6. **Repeat** 2–5 until $E(\vec{\theta})$ converges.



Why this fits superconducting qubit hardware

- The ansatz $\hat{U}(\vec{\theta})$ is built from **single-qubit rotations** (calibrated via Rabi as in §16.5) and **two-qubit entanglers** (CZ, iSWAP — exchange interaction from the qubit-qubit coupling of §15).
- Each Pauli expectation $\langle \hat{P}_k \rangle$ is obtained by **dispersive readout** (§16.4) after a basis-changing layer.
- Short circuit depth $\Rightarrow$ stays within T_2^* of a transmon (a few μs).

Strengths and limitations

Strength	Limitation
Shallow circuits, NISQ-friendly	Optimizer can get stuck in barren plateaus (gradient ≈ 0 for random initializations)
Variational bound $E \geq E_0$ guarantees no overshoot	Number of Pauli terms scales $\mathcal{O}(n^4)$ for chemistry $\rightarrow$ many measurements
Robust to systematic gate errors (they shift $\vec{\theta}^*$, not E_0)	Choice of ansatz controls expressibility vs trainability
Hybrid loop offloads hard step to classical CPU	No theoretical guarantee of polynomial convergence

Worked toy example — H_2 in minimal basis

After Jordan–Wigner mapping, the H_2 Hamiltonian at bond length R reduces (with symmetry tapering) to a **2-qubit** operator

$$\hat{H} = g_0 I + g_1 Z_0 + g_2 Z_1 + g_3 Z_0 Z_1 + g_4 X_0 X_1 + g_5 Y_0 Y_1,$$

with coefficients $g_i(R)$ tabulated. A 1-parameter ansatz $|\psi(\theta)\rangle = \cos(\theta/2)|01\rangle - \sin(\theta/2)|10\rangle$ already reproduces the dissociation curve to chemical accuracy — first demonstrated by Peruzzo et al. (2014) and O’Malley et al. (2016) on superconducting hardware.

How VQE connects to the rest of the course

- **State preparation** = the $\pi/2$ / π pulses calibrated by Rabi (§16.5).
- **Coupling required for ansatz entanglers** = the qubit–qubit exchange Hamiltonian (§15, end).
- **Energy readout** = repeated dispersive measurements (§16.4), each shot projecting onto σ_z .
- **The Hamiltonian whose ground state we hunt** can itself be a circuit Hamiltonian (e.g. ground state of a transmon array, lattice gauge model) — the same physics from §13–§14 reappears as the target problem.