

PHYS 514
Practice Questions Set – 4

Q.1 Quadrature Operator

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 = \frac{i}{2}(\hat{a} - \hat{a}^\dagger)$$

These new operators represent the phase-space coordinates for the quantum field, often called the “quadrature” operators (for historical reasons). They also define the coordinates of a 2D pictorial phase-space representation of the quantum oscillator mode.

(a) Show that

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

(b) Consider a system in the vacuum or ground state, $|\psi\rangle = |0\rangle$, show that

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0$$

$$\Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{1}{4}$$

(c) Consider a system in the Fock state, $|\psi\rangle = |n\rangle$, show that

$$\langle \hat{X}_1 \rangle = \langle \hat{X}_2 \rangle = 0$$

$$\Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{n}{2} + \frac{1}{4}$$

(d) Consider a system in the Coherent state, $|\psi\rangle = |\alpha\rangle$, show that

$$\langle \hat{X}_1 \rangle = |\alpha| \cos \theta$$

$$\langle \hat{X}_2 \rangle = |\alpha| \sin \theta$$

$$\Delta \hat{X}_1^2 = \Delta \hat{X}_2^2 = \frac{1}{4}$$

Q.2 Rabi Oscillations

A Rabi experiment describes the evolution of a qubit when its two energy levels are coupled together by an interaction Hamiltonian. Time evolution unitary operator U that defines the time evolution of Rabi oscillation is given by

$$U(t) = \begin{pmatrix} \cos \frac{1}{2} \Omega t & -i \sin \frac{1}{2} \Omega t \\ -i \sin \frac{1}{2} \Omega t & \cos \frac{1}{2} \Omega t \end{pmatrix}$$

(a) Given the initial state $|\psi\rangle = |0\rangle$, hence show that the output state after time t is:

$$|\psi(t)\rangle = \begin{pmatrix} \cos \frac{1}{2} \Omega t \\ -i \sin \frac{1}{2} \Omega t \end{pmatrix}$$

and that the ground and excited state populations are:

$$p_0(t) = \frac{1}{2} (1 + \cos \Omega t)$$

$$p_1(t) = \frac{1}{2} (1 - \cos \Omega t)$$

(b) Sketch the ground state population as a function of delay time, t.

Q.3 Quantum LC oscillator's wave function

Following is the ground state of LC oscillator

$$\Psi_0(\phi) = \frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2}$$

- (a) Using raising operator find the expression for one photon $\Psi_1(\phi)$ and two photon $\Psi_2(\phi)$ states
 (b) Plot the $\Psi_+(\phi)$ versus ϕ , where

$$\Psi_+(\phi) = \frac{1}{\sqrt{2}} (\Psi_0(\phi) + \Psi_1(\phi))$$

Solution

$$\hat{\phi} = \Phi_{ZPF} (\hat{a} + \hat{a}^\dagger)$$

$$\hat{a}^\dagger = \frac{\hat{\phi}}{\Phi_{ZPF}} - \hat{a}$$

$$\hat{a}^\dagger \Psi_0(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} - \hat{a} \right) \Psi_0(\phi)$$

$$\Psi_1(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} \Psi_0(\phi) - \hat{a} \Psi_0(\phi) \right)$$

$$\hat{a} \Psi_0(\phi) = 0$$

Therefore

$$\Psi_1(\phi) = \frac{\hat{\phi}}{\Phi_{ZPF}} \frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2}$$

Now

$$\Psi_2(\phi) = \hat{a}^\dagger \Psi_1(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} - \hat{a} \right) \Psi_1(\phi)$$

$$\Psi_2(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} \Psi_1(\phi) - \hat{a} \Psi_1(\phi) \right)$$

$$\Psi_2(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} \Psi_1(\phi) - \Psi_0(\phi) \right)$$

$$\Psi_2(\phi) = \left(\frac{\hat{\phi}}{\Phi_{ZPF}} \left[\frac{\hat{\phi}}{\Phi_{ZPF}} \frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} \right] - \frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} \right)$$

$$\Psi_2(\phi) = \left[\left(\frac{\hat{\phi}}{\Phi_{ZPF}} \right)^2 - 1 \right] \left(\frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} \right)$$

Now find the expression for

$$\Psi_+(\phi) = \frac{1}{\sqrt{2}} (\Psi_0(\phi) + \Psi_1(\phi))$$

$$\Psi_+(\phi) = \frac{1}{\sqrt{2}} \left[\frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} + \frac{\hat{\phi}}{\Phi_{ZPF}} \frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} \right]$$

$$\Psi_+(\phi) = \frac{1}{\sqrt{2}} \left(1 + \frac{\hat{\phi}}{\Phi_{ZPF}} \right) \left(\frac{1}{[2\pi\Phi_{ZPF}^2]^{1/4}} e^{-\frac{1}{4}} \frac{\phi^2}{\Phi_{ZPF}^2} \right)$$