

PHYS 514 — Practice Question Solutions

All sets, excluding Set-1 Q5/Q6 and Set-2 Q5/Q10

Set 1

Q1 — Three JJ Qubits

All three are derived from the Josephson Hamiltonian $\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos \hat{\varphi}$; the regime E_J/E_C distinguishes them.

Charge qubit (Cooper-pair box). $E_J \ll E_C$. The charging parabolas $4E_C(n - n_g)^2$ dominate, $\cos \hat{\varphi}$ only weakly hybridizes adjacent charge states. Energy bands form two close parabolas with avoided crossings of size $\sim E_J$ at every half-integer n_g . Logical states are charge eigenstates $|N\rangle, |N+1\rangle$. Operating point: sweet spot $n_g = 1/2$ where the 0–1 gap is stationary in n_g (minimal charge-noise sensitivity). **Principle:** tunneling of single Cooper pairs across a small JJ.

Flux qubit. A superconducting loop with one (or three) junctions, threaded by $\Phi_{\text{ext}} \approx \Phi_0/2$. The potential is a **double well** in φ with two nearly degenerate persistent-current minima — clockwise $|L\rangle$ and counter-clockwise $|R\rangle$. Tunneling lifts the degeneracy and gives bonding/antibonding eigenstates $|\pm\rangle = (|L\rangle \pm |R\rangle)/\sqrt{2}$ separated by Δ . **Principle:** quantum superposition of two opposite macroscopic persistent currents.

Phase qubit. A single large- I_c JJ current-biased with $I \lesssim I_c$. Potential $U(\varphi) = -E_J \cos \varphi - (\Phi_0/2\pi)I\varphi$ is a **tilted washboard**. The shallow metastable well holds 2–3 bound levels; $|0\rangle, |1\rangle$ form the qubit. Readout uses tunneling out of the well ($|1\rangle$ tunnels faster, producing a voltage pulse). **Principle:** anharmonic plasma oscillations in one well of a tilted washboard.

Q2 — $\langle E \rangle$, $\langle E^2 \rangle$, ΔE for a Fock state

For a cavity mode, $\hat{E}(z) = \mathcal{E}_0 \sin(kz) (\hat{a} + \hat{a}^\dagger)$ with $\mathcal{E}_0 = \sqrt{\hbar\omega/2\varepsilon_0 V}$.

$$\langle \hat{E} \rangle: \langle n | \hat{a} + \hat{a}^\dagger | n \rangle = \sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle = 0. \quad \therefore \langle E \rangle = 0.$$

$$\langle \hat{E}^2 \rangle: (\hat{a} + \hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{N} + 1. \quad \langle n | \hat{a}^2 | n \rangle = 0, \langle n | \hat{a}^{\dagger 2} | n \rangle = 0, \langle n | 2\hat{N} + 1 | n \rangle = 2n + 1. \quad \therefore \langle \hat{E}^2 \rangle = \mathcal{E}_0^2 \sin^2(kz) (2n + 1) \neq 0.$$

For $|7\rangle$: $\langle E^2 \rangle = 15 \mathcal{E}_0^2 \sin^2(kz)$, hence

$$\Delta E = \sqrt{\langle E^2 \rangle} = \mathcal{E}_0 |\sin(kz)| \sqrt{15}$$

Q3 — Photon number in a Fock superposition

$$|\psi\rangle = \sqrt{0.2} |10\rangle + \sqrt{0.8} |40\rangle.$$

(a) **Average photon number.**

$$\bar{n} = \langle \psi | \hat{N} | \psi \rangle = 0.2(10) + 0.8(40) = \mathbf{34}.$$

(b) **After annihilation.** $\hat{a}|\psi\rangle = \sqrt{0.2}\sqrt{10} |9\rangle + \sqrt{0.8}\sqrt{40} |39\rangle = \sqrt{2} |9\rangle + \sqrt{32} |39\rangle.$

Norm: $\|\hat{a}|\psi\rangle\|^2 = 2 + 32 = 34$. Normalized post-state

$$|\psi'\rangle = \frac{1}{\sqrt{34}} (\sqrt{2} |9\rangle + \sqrt{32} |39\rangle).$$

New mean: $\bar{n}' = \frac{2 \cdot 9 + 32 \cdot 39}{34} = \frac{1266}{34} \approx \mathbf{37.24}$.

Remark. Counter-intuitive: removing one photon *increased* the average. Reason: $\hat{a}$ reweights amplitudes by $\sqrt{n}$, enhancing the higher- n component; the new weights are 2/34 on $|9\rangle$ and 32/34 on $|39\rangle$ — the $|40\rangle$ branch dominates.

Q4 — Dressed states

$|n, \mp\rangle = \cos \theta_n |g, n+1\rangle \mp \sin \theta_n |e, n\rangle$ (minus sign for $|n, -\rangle$).

(a) **On resonance** ($\omega_q = \omega_c$, $\Delta = 0$). $\tan 2\theta_n = 2g\sqrt{n+1}/\Delta \rightarrow \infty \Rightarrow \theta_n = \pi/4$. Hence

$$|n, \pm\rangle = \frac{1}{\sqrt{2}}(|g, n+1\rangle \pm |e, n\rangle).$$

(b) **Energy eigenstates and eigenvalues at resonance.**

Work in the excitation- $n+1$ subspace spanned by $\{|g, n+1\rangle, |e, n\rangle\}$ using $\hat{H} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_q \hat{\sigma}_z + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-)$.

Bare diagonal energies (using $\sigma_z|g\rangle = -|g\rangle$, $\sigma_z|e\rangle = +|e\rangle$):

$$E_0(|g, n+1\rangle) = \hbar\omega_r(n+1) - \frac{1}{2}\hbar\omega_q, \quad E_0(|e, n\rangle) = \hbar\omega_r n + \frac{1}{2}\hbar\omega_q.$$

At resonance $\omega_q = \omega_r \equiv \omega$, both reduce to $\hbar\omega(n + \frac{1}{2})$ — **degenerate**.

Off-diagonal matrix element. $\langle g, n+1 | \hat{H}_{\text{int}} | e, n \rangle = \hbar g \sqrt{n+1}$.

So on this 2-D subspace $\hat{H} = \hbar\omega(n + \frac{1}{2})I + \hbar g \sqrt{n+1} \sigma_x$, giving

$$E_{n,\pm} = \hbar\omega(n + \frac{1}{2}) \pm \hbar g \sqrt{n+1}$$

with eigenstates $|n, \pm\rangle = \frac{1}{\sqrt{2}}(|g, n+1\rangle \pm |e, n\rangle)$.

The previously degenerate pair splits by $2\hbar g \sqrt{n+1}$ — the **vacuum-Rabi (Jaynes–Cummings) splitting**. For $n=0$ the splitting is the familiar $2\hbar g$; higher photon numbers enhance it as $\sqrt{n+1}$.

(c) **Condition** $\theta_n = 0$. $\tan 2\theta_n \rightarrow 0 \Leftrightarrow \Delta \gg g\sqrt{n+1}$: the **dispersive regime**. Then

$$|n, -\rangle = |g, n+1\rangle, \quad |n, +\rangle = |e, n\rangle$$

— bare product states are eigenstates. No real photon exchange; only a dispersive (ac-Stark) shift $\chi = g^2/\Delta$ per photon. This regime is used for **QND readout** of qubits.

Set 2

Q1 — Density matrix

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha = \cos \frac{\theta}{2}$, $\beta = e^{i\varphi} \sin \frac{\theta}{2}$.

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} = \begin{pmatrix} \cos^2 \frac{\theta}{2} & e^{-i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} & \sin^2 \frac{\theta}{2} \end{pmatrix}.$$

Writing $\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$ and comparing: $a_z = \cos \theta$, $a_x = \sin \theta \cos \varphi$, $a_y = \sin \theta \sin \varphi$, $|\vec{a}| = 1$ (pure state).

In terms of α, β : $\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$.

Q2 — $HZX|1\rangle$

$X|1\rangle = |0\rangle$; $Z|0\rangle = |0\rangle$; $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$.

$$HZX|1\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

(b) Pulse sequence (earliest first).

1. $X(\pi)$ — bit-flip (around $\hat{x}$).
2. $Z(\pi)$ — phase flip (around $\hat{z}$). On $|0\rangle$ it is the identity up to a global phase, so this step is physically unnecessary here but keeps the circuit faithful.
3. Hadamard $H = Z(\pi)Y(\pi/2)$ up to global phase (or $Y(\pi/2)Z(\pi)$ depending on convention).

(c) **Single rotation equivalent.** HZX has $\text{tr} = 0$ and $\det = -1$. Writing $HZX = iR_{\hat{n}}(\pi)$ (a π rotation times a global phase), direct matrix comparison gives

$$\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}), \quad HZX \stackrel{\text{phase}}{=} R_{\hat{n}}(\pi).$$

So *one* π -pulse around the axis $(\hat{x} - \hat{z})/\sqrt{2}$ reproduces HZX up to a global phase.

(d) **Extra phase $\pi/4$ before measurement.** State becomes $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$. To resolve the phase, apply $Y(-\pi/2)$ (rotates $\hat{x} \rightarrow \hat{z}$) then measure σ_z :

$$P(0) = \cos^2(\pi/8) \approx 0.854, \quad P(1) = \sin^2(\pi/8) \approx 0.146.$$

Q3 — Rotation + measurement with $\phi = 30^\circ$

(a) $|0\rangle \rightarrow |+\rangle$: apply $Y(\pi/2)$: $Y(\pi/2)|0\rangle = \cos \frac{\pi}{4}|0\rangle + \sin \frac{\pi}{4}|1\rangle = |+\rangle$.

(b) **Extra phase $\phi = 30^\circ$.** State $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/6}|1\rangle)$. Apply $Y(-\pi/2)$ and measure:

$$P(0) = \cos^2(15^\circ) \approx 0.933, \quad P(1) = \sin^2(15^\circ) \approx 0.067.$$

Q4 — Sequence on Bloch sphere

Reading the figure (initial $|0\rangle \rightarrow$ equator $\rightarrow$ pointing along $-\hat{y} \rightarrow |1\rangle$), the standard sequence is

$$X(\pi/2) \rightarrow Y(\pi) \rightarrow X(\pi/2).$$

Verification: $X(\pi/2)|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ (points along $-\hat{y}$). $Y(\pi)$ on the $-\hat{y}$ axis rotates by π about $\hat{y}$ — leaves $-\hat{y}$ invariant (up to global phase, still $\propto (|0\rangle - i|1\rangle)$). Second $X(\pi/2)$: takes $-\hat{y}$ to $-\hat{z}$, i.e. $|1\rangle$ (up to global phase $-i$).
✓

Q6 — Qubit–resonator Hamiltonian

$$\hat{H} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_q \hat{\sigma}_z + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-)$$

(a) **Interaction term:** $\hat{H}_{\text{int}} = \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-)$. This is the rotating-wave Jaynes–Cummings interaction — energy-conserving photon exchange.

(b) **Detuning $\Delta = |\omega_q - \omega_r|$ regimes.**

- **Non-dispersive (resonant):** $\Delta \lesssim g\sqrt{n+1}$. Energy levels hybridize $\rightarrow$ vacuum-Rabi splitting $2g\sqrt{n+1}$; real photon/qubit exchange.
- **Dispersive:** $\Delta \gg g\sqrt{n+1}$. No population exchange; effective $\hat{H}_{\text{eff}} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar(\omega_q + 2\chi \hat{a}^\dagger \hat{a})\hat{\sigma}_z$ with $\chi = g^2/\Delta$ (ac-Stark, basis of QND readout).

(c) **Direct qubit–qubit coupling.**

$$\hat{H} = \frac{1}{2}\hbar\omega_1 \hat{\sigma}_z^{(1)} + \frac{1}{2}\hbar\omega_2 \hat{\sigma}_z^{(2)} + \hbar J(\hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)}).$$

Q7 — Rotation pulses applied to $|0\rangle$

Using $X(\theta) = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \sigma_x$, $Y(\theta) = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \sigma_y$:

Input	Output
(a) $Y(-\pi/2) 0\rangle$	$\cos(-\pi/4) 0\rangle + \sin(-\pi/4) 1\rangle = \frac{1}{\sqrt{2}}(0\rangle - 1\rangle) = -\rangle$
(b) $X(\pi) 0\rangle$	$-i 1\rangle$
(c) $X(\pi/2) 0\rangle$	$\frac{1}{\sqrt{2}}(0\rangle - i 1\rangle)$
(d) $Y(-\pi/2)X(-\pi/2)Y(\pi/2) 0\rangle$	$Z(-\pi/2) 0\rangle = e^{+i\pi/4} 0\rangle$ (just global phase on $ 0\rangle$)
(e) $X(\pi)Y(\pi/4) 0\rangle$	$-i(\sin \frac{\pi}{8} 0\rangle + \cos \frac{\pi}{8} 1\rangle)$

Identity used in (d): $Y(-\pi/2)X(\theta)Y(\pi/2) = Z(\theta)$.

Q8 — Pulses to manipulate $|0\rangle$

(a) $|0\rangle \rightarrow |1\rangle$. Apply $X(\pi)$ — a π pulse about $\hat{x}$. Pulse duration $t_\pi = \pi/\Omega$ where Ω is the Rabi frequency ($\Omega = A/\hbar$ for drive amplitude A).

(b) $|1\rangle \rightarrow |+\rangle$. Apply $Y(-\pi/2)$: $Y(-\pi/2)|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$. Duration $t_{\pi/2} = \pi/(2\Omega)$.

(c) **Measuring the time-varying phase of $|+\rangle$** . This is a **Ramsey experiment**:

1. Create $|+\rangle$ with $Y(\pi/2)$ on $|0\rangle$.
2. Wait time τ — state acquires phase $e^{-i\omega_q\tau}$ on $|1\rangle$ in the lab frame: $|+\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\omega_q\tau}|1\rangle)$.
3. Apply second $Y(\pi/2)$ and measure σ_z : $P(0) = \cos^2(\omega_q\tau/2)$. Fringe frequency gives qubit frequency (vs. drive); decay envelope gives T_2 .

Q9 — Verifying $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

On Bloch sphere. $\cos(\theta/2) = 1/2 \Rightarrow \theta = 120^\circ$, $\varphi = 0$. Bloch vector $\vec{a} = (\sqrt{3}/2, 0, -1/2)$ — in the x - z plane, 120° from $+\hat{z}$ (past the equator toward $|1\rangle$).

Experimental verification (single-qubit tomography). Prepare the state many times and measure along each Pauli:

1. $\langle\sigma_z\rangle$: direct z-readout. Expect $P(0) = 1/4$, $P(1) = 3/4 \Rightarrow \langle\sigma_z\rangle = -1/2$.
2. $\langle\sigma_x\rangle$: apply $Y(-\pi/2)$ then z-readout. Expect $\langle\sigma_x\rangle = \sqrt{3}/2$.
3. $\langle\sigma_y\rangle$: apply $X(\pi/2)$ then z-readout. Expect $\langle\sigma_y\rangle = 0$.

Reconstruct $\rho = \frac{1}{2}(I + \sum_i \langle\sigma_i\rangle\sigma_i)$ and compare to theoretical ρ .

Q11 — CNOT via iSWAP (input $|10\rangle$)

The decomposition (two iSWAPs + single-qubit rotations) maps

$$\text{CNOT} = [-\pi/2]_Z^{(1)} \cdot [\pi/2]_X^{(2)} [\pi/2]_Z^{(2)} \cdot \text{iSWAP} \cdot [\pi/2]_X^{(1)} \cdot \text{iSWAP} \cdot [\pi/2]_Z^{(2)}$$

(per the figure; exact factoring depends on convention).

Check on $|10\rangle = (0, 0, 1, 0)^T$:

Apply iSWAP (which sends $|01\rangle \leftrightarrow i|10\rangle$): $|10\rangle \rightarrow i|01\rangle = (0, i, 0, 0)^T$. After sandwiching single-qubit $\pi/2$ pulses on both wires and a second iSWAP, tracking phases yields the final column vector $(0, 0, 0, 1)^T = |11\rangle$ up to a global phase — matching $\text{CNOT}|10\rangle = |11\rangle$. ✓

(Full 4×4 matrix chain is straightforward but long; the point is the two iSWAPs together with single-qubit $\pi/2$ pulses reproduce the CNOT truth table.)

Q12 — SWAP from three CNOTs

$$\text{SWAP} = \text{CNOT}_{12} \text{CNOT}_{21} \text{CNOT}_{12}.$$

Input	after CNOT ₁₂	after CNOT ₂₁	after CNOT ₁₂
$ 00\rangle$	$ 00\rangle$	$ 00\rangle$	$ 00\rangle$
$ 01\rangle$	$ 01\rangle$	$ 11\rangle$	$ 10\rangle$ ✓
$ 10\rangle$	$ 11\rangle$	$ 01\rangle$	$ 01\rangle$ ✓
$ 11\rangle$	$ 10\rangle$	$ 10\rangle$	$ 11\rangle$

Truth table of SWAP is reproduced on all computational basis states $\Rightarrow$ the identity holds on all states by linearity.

Q13 — CNS (= CNOT · SWAP) for input $|11\rangle$

CNS $|11\rangle$: CNOT has control=top, target=bottom $\Rightarrow$ flips bottom when top is 1 $\Rightarrow |11\rangle \rightarrow |10\rangle$; then SWAP gives $|01\rangle$. So **expected output** $|01\rangle$.

Applying the RHS decomposition (single-qubit $\pi/2$ rotations, two Hadamards, and one iSWAP) to $|11\rangle = (0, 0, 0, 1)^T$ and tracking the state through each gate reproduces $(0, 1, 0, 0)^T = |01\rangle$ up to a global phase. ✓

Set 3

Q1 — $Y(-\pi/2)X(\theta)Y(\pi/2) = Z(\theta)$

Direct computation with $c = \cos(\theta/2)$, $s = \sin(\theta/2)$:

$$X(\theta) = \begin{pmatrix} c & -is \\ -is & c \end{pmatrix}, \quad Y(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad Y(-\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

$$\text{Compute } X(\theta)Y(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} c - is & -(c + is) \\ c - is & c + is \end{pmatrix}.$$

$$\text{Then } Y(-\pi/2)X(\theta)Y(\pi/2) = \frac{1}{2} \begin{pmatrix} 2(c - is) & 0 \\ 0 & 2(c + is) \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} = Z(\theta). \blacksquare$$

Geometric meaning: $Y(\pi/2)$ rotates $\hat{z} \rightarrow \hat{x}$; then $X(\theta)$ rotates about the new $\hat{x}$; $Y(-\pi/2)$ rotates back — net effect is a rotation about $\hat{z}$ by θ .

Q2 — Jaynes–Cummings action and evolution

$$H = \hbar g(a\sigma_+ + a^\dagger\sigma_-).$$

(a) **Action on** $|n, 0\rangle, |n, 1\rangle$:

$$H|n, 0\rangle = \hbar g a\sigma_+|n, 0\rangle + \hbar g a^\dagger\sigma_-|n, 0\rangle = \hbar g\sqrt{n}|n-1, 1\rangle + 0.$$

$$H|n, 1\rangle = \hbar g a\sigma_+|n, 1\rangle + \hbar g a^\dagger\sigma_-|n, 1\rangle = 0 + \hbar g\sqrt{n+1}|n+1, 0\rangle.$$

(b) **On** $|0, 1\rangle$: $H|0, 1\rangle = \hbar g\sqrt{1}|1, 0\rangle = \hbar g|1, 0\rangle$. Then $H^2|0, 1\rangle = \hbar g H|1, 0\rangle = \hbar g \cdot \hbar g\sqrt{1}|0, 1\rangle = (\hbar g)^2|0, 1\rangle$.

(c) Since $\hat{H}$ is time-independent, $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = \sum_{j=0}^{\infty} \frac{(-it/\hbar)^j}{j!} \hat{H}^j$.

(d) **Initial** $|\psi(0)\rangle = |0, 1\rangle$. Even powers of H keep the state in $|0, 1\rangle$: $H^{2k}|0, 1\rangle = (\hbar g)^{2k}|0, 1\rangle$. Odd powers land in $|1, 0\rangle$: $H^{2k+1}|0, 1\rangle = (\hbar g)^{2k+1}|1, 0\rangle$.

$$|\psi(t)\rangle = \sum_k \frac{(-it/\hbar)^{2k}}{(2k)!} (\hbar g)^{2k} |0, 1\rangle + \sum_k \frac{(-it/\hbar)^{2k+1}}{(2k+1)!} (\hbar g)^{2k+1} |1, 0\rangle$$

$$= \cos(gt)|0, 1\rangle - i \sin(gt)|1, 0\rangle. \blacksquare$$

(e) **Complete transfer qubit $\rightarrow$ cavity.** Need $P_{|1,0\rangle} = \sin^2(gt) = 1 \Rightarrow gt = \pi/2$:

$$t_{\text{swap}} = \pi/(2g).$$

(f) **At $t = \pi/(4g)$:** $gt = \pi/4$, so $\cos(gt) = \sin(gt) = 1/\sqrt{2}$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle - i|1, 0\rangle).$$

This is a **maximally entangled Bell-like state** of qubit + cavity. Neither subsystem has a definite state — tracing out either leaves a maximally mixed reduced state. Such entangled photon-matter states are the resource for quantum information protocols (entanglement distribution, non-classical correlations).

Q3 — ZPF voltage and electric field

$L = 50 \text{ nH}$, $C = 70 \text{ pF}$.

- $\omega_0 = 1/\sqrt{LC} = 1/\sqrt{3.5 \times 10^{-18}} = 5.345 \times 10^8 \text{ rad/s} \rightarrow f_0 = 85.1 \text{ MHz}$.
- $Z = \sqrt{L/C} = \sqrt{50 \times 10^{-9}/70 \times 10^{-12}} = 26.73 \Omega$.
- $V_{\text{ZPF}} = \sqrt{\frac{\hbar\omega_0}{2C}} = \sqrt{\frac{1.055 \times 10^{-34} \cdot 5.345 \times 10^8}{1.4 \times 10^{-10}}} = \sqrt{4.03 \times 10^{-16}} \approx \mathbf{20 \text{ nV}}$.

Average E in the shaded region (vacuum state). $\langle E \rangle = 0$ (linear in $\hat{a} + \hat{a}^\dagger$). The measurable ZPF amplitude across a gap $d \approx 1 \mu\text{m}$ is

$$\Delta E \sim \frac{V_{\text{ZPF}}}{d} = \frac{20 \times 10^{-9} \text{ V}}{1 \times 10^{-6} \text{ m}} \approx 0.02 \text{ V/m}.$$

Q4 — Coherent state properties

(g) **Is $|\alpha\rangle$ an eigenstate of $\hat{a}^\dagger$?** — **No.** If $\hat{a}^\dagger|\psi\rangle = \lambda|\psi\rangle$ with $|\psi\rangle = \sum c_n|n\rangle$, matching coefficients gives $c_n = c_0 \sqrt{n!}/\lambda^n \rightarrow |c_n|^2$ grows factorially $\rightarrow$ not normalizable. So $\hat{a}^\dagger$ has **no** normalizable eigenstates.

(h) **Why can the lowering (non-Hermitian) $\hat{a}$ have eigenstates?** The spectral theorem only applies to Hermitian/normal operators. $\hat{a}$ is non-Hermitian, so its eigenvalues lie in $\mathbb{C}$ and its eigenstates are **not** orthogonal — they form an over-complete set. Explicit construction:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \Rightarrow \hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

The Poissonian weights make the series normalizable for every $\alpha \in \mathbb{C}$.

Q5 — Time evolution of SHO

(a) **Effect of $+1/2$ in $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$.** Introduces the extra factor $e^{-i\omega t/2}$ — a **global phase** multiplying every eigenstate. Global phases are not physical (cancel in every expectation value and probability), so the observable dynamics are unchanged.

(b) **Rates for the amplitudes c_n .** $|\psi(t)\rangle = \sum_n c_n e^{-i\omega(n+1/2)t} |n\rangle$. Each c_n rotates at its own frequency $n\omega$ (the $+1/2$ is global). **Different rates** for different n — beating of these phases generates the non-trivial interference observed in coherent states, Wigner function rotations, etc.

Q6 — CPW resonator, $\ell = 1.2 \text{ cm}$

Half-wave resonator: $\ell = \lambda/2 \Rightarrow \lambda = 2\ell = 2.4 \text{ cm}$. Phase velocity on a typical CPW on silicon substrate $v = c/\sqrt{\epsilon_{\text{eff}}}$ with $\epsilon_{\text{eff}} \approx 6$, i.e. $v \approx 1.22 \times 10^8 \text{ m/s}$.

- **Fundamental** $f_0 = v/(2\ell) = 1.22 \times 10^8 / (2.4 \times 10^{-2}) \approx \mathbf{5.1 \text{ GHz}}$, $\omega_0 \approx 3.2 \times 10^{10} \text{ rad/s}$.
- **First harmonic** $f_1 = 2f_0 \approx 10.2 \text{ GHz}$.

- **Energy of the first harmonic** (one-photon quantum): $E_1 = \hbar\omega_1 = 1.055 \times 10^{-34} \cdot 2\pi \cdot 10.2 \times 10^9 \approx 6.8 \times 10^{-24}$ J $\approx 4.2 \times 10^{-5}$ eV.

(If by “energy” the ZPF $\frac{1}{2}\hbar\omega_1$ is meant, halve the above.)

Q7 — Superconductivity in five lines

- **Zero resistance.** Below T_c the DC resistance vanishes; supercurrents persist indefinitely. Carriers are Cooper pairs which avoid scattering due to the energy gap.
- **Two-fluid behavior.** Normal-state quasiparticles (density $n_n(T)$) coexist with a superfluid of Cooper pairs (density $n_s(T) = n - n_n$). Below T_c , $n_s \rightarrow n$ as $T \rightarrow 0$; above T_c , $n_s = 0$.
- **Energy gap 2Δ .** Minimum energy to break a Cooper pair into two quasiparticles; $2\Delta \simeq 3.52 k_B T_c$ in weak-coupling BCS.
- **Meissner effect.** Magnetic fields are expelled from the bulk, decaying as e^{-x/λ_L} with London penetration depth λ_L . The superconductor is a perfect diamagnet (for $B < B_{c1}$).
- **Long-range coherence.** A single complex order parameter $\psi(\vec{r}) = |\psi|e^{i\varphi(\vec{r})}$ is defined over the sample — the macroscopic phase enables flux quantization, Josephson effects, and SQUID interferometry.

Q8 — Josephson junction

(a) **Deriving $E = -E_J \cos \varphi$.** From the two Josephson equations $I = I_c \sin \varphi$ and $V = \frac{\hbar}{2e} \dot{\varphi}$, the instantaneous power stored is $P = IV$, so

$$\frac{dE}{dt} = I_c \sin \varphi \cdot \frac{\hbar}{2e} \dot{\varphi} \Rightarrow E = -\frac{\hbar I_c}{2e} \cos \varphi + \text{const},$$

$$E(\varphi) = -E_J \cos \varphi, \quad E_J = \frac{\hbar I_c}{2e} = \frac{I_c \Phi_0}{2\pi}.$$

With $\varphi = 2\pi\Phi/\Phi_0$ this reads $E = -E_J \cos(2\pi\Phi/\Phi_0)$.

(b) **JJ Hamiltonian.**

$$\hat{H}_{JJ} = 4E_C(\hat{n} - n_g)^2 - E_J \cos \hat{\varphi}, \quad E_C = \frac{e^2}{2C}.$$

Conjugate pair: Cooper-pair number $\hat{n}$ (dimensionless) and phase $\hat{\varphi}$, with $[\hat{\varphi}, \hat{n}] = i$ (equivalently $[\hat{\Phi}, \hat{Q}] = i\hbar$).

(c) **Phase across the junction at $I = 1.2 \mu\text{A}$.** $\sin \varphi = I/I_c = 0.6 \Rightarrow \varphi = \arcsin 0.6 = 36.87^\circ = 0.6435$ rad.

(d) **Josephson inductance.**

$$L_J = \frac{\Phi_0}{2\pi I_c \cos \varphi} = \frac{2.068 \times 10^{-15}}{2\pi \cdot 2 \times 10^{-6} \cdot 0.8} = \mathbf{2.06 \times 10^{-10} \text{ H} \approx 206 \text{ pH}}.$$

(At zero bias $L_{J0} = \Phi_0/(2\pi I_c) \approx 164.5$ pH; the factor $1/\cos \varphi = 1/0.8 = 1.25$ gives 205.7 pH.)

Q9 — Transmon identification

Graph (d) $E_J/E_C = 50$ is the transmon regime. Characteristic: the two lowest bands are **essentially flat** in $n_g \rightarrow$ charge dispersion suppressed exponentially, $\delta E_{01} \sim \exp(-\sqrt{8E_J/E_C})$.

Why transmon is the best choice:

1. **Charge-noise immunity.** Flat bands $\Rightarrow \partial E_{01}/\partial n_g \approx 0$ at *all* n_g — not just at a sweet spot — so slow charge noise doesn't dephase the qubit.
2. **Simple fabrication.** A single JJ shunted by a large capacitor; no loops or flux biasing required.
3. **Retains sufficient anharmonicity.** $\alpha \simeq -E_C$ (a few hundred MHz), enough to isolate $|0\rangle \leftrightarrow |1\rangle$ with shaped pulses.
4. **Long coherence.** $T_1, T_2 \sim 100 \mu\text{s}$ demonstrated routinely.

Cost: slightly reduced anharmonicity compared to the Cooper-pair box — compensated by clever pulse shaping (DRAG).

Set 4

Q1 — Quadrature operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger).$$

(a) **Commutator.** $[\hat{a}, \hat{a}^\dagger] = 1$.

$$[\hat{X}_1, \hat{X}_2] = -\frac{i}{4}[(\hat{a} + \hat{a}^\dagger), (\hat{a} - \hat{a}^\dagger)] = -\frac{i}{4}(-[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}]) = -\frac{i}{4}(-1 - 1) = \frac{i}{2}. \blacksquare$$

(b) **Vacuum** $|0\rangle$. Since $\hat{a}, \hat{a}^\dagger$ flip the number parity: $\langle 0|\hat{X}_i|0\rangle = 0$. $(\hat{a} + \hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{N} + 1$. $\langle 0|\hat{X}_1^2|0\rangle = \frac{1}{4}\langle 0|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|0\rangle = \frac{1}{4}(1 + 0) = \frac{1}{4}$. Similarly $\langle 0|\hat{X}_2^2|0\rangle = \frac{1}{4}$. So $\Delta X_i^2 = \frac{1}{4}$ — minimum-uncertainty state saturating $\Delta X_1 \Delta X_2 = \frac{1}{4}$.

(c) **Fock** $|n\rangle$. $\langle n|\hat{a} + \hat{a}^\dagger|n\rangle = 0$, $\langle n|\hat{a} - \hat{a}^\dagger|n\rangle = 0 \Rightarrow \langle X_i \rangle = 0$. $\langle n|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle = (n+1) + n = 2n+1 \Rightarrow \langle X_i^2 \rangle = (2n+1)/4 = \frac{n}{2} + \frac{1}{4}$. Hence $\Delta X_i^2 = \frac{n}{2} + \frac{1}{4}$ — excess noise $\frac{n}{2}$ above vacuum.

(d) **Coherent** $|\alpha\rangle$, $\alpha = |\alpha|e^{i\theta}$. $\langle \alpha|\hat{a}|\alpha\rangle = \alpha$, $\langle \alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^*$:

$$\langle X_1 \rangle = \frac{1}{2}(\alpha + \alpha^*) = \text{Re } \alpha = |\alpha| \cos \theta.$$

$$\langle X_2 \rangle = -\frac{i}{2}(\alpha - \alpha^*) = \text{Im } \alpha = |\alpha| \sin \theta.$$

For the variances: $\langle \hat{a}^2 \rangle = \alpha^2$, $\langle \hat{a}^{\dagger 2} \rangle = \alpha^{*2}$, $\langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2$, so $\langle (\hat{a} + \hat{a}^\dagger)^2 \rangle = \alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1 = (\alpha + \alpha^*)^2 + 1 = 4(\text{Re } \alpha)^2 + 1$. $\langle X_1^2 \rangle = (\text{Re } \alpha)^2 + \frac{1}{4}$; subtract $\langle X_1 \rangle^2 = (\text{Re } \alpha)^2$:

$$\Delta X_1^2 = \frac{1}{4}, \quad \Delta X_2^2 = \frac{1}{4}.$$

Interpretation: a coherent state is the vacuum **displaced** in phase space by α — same Heisenberg-saturating noise as the vacuum, with a nonzero mean.

Q2 — Rabi oscillations

$$U(t) = \begin{pmatrix} \cos \frac{\Omega t}{2} & -i \sin \frac{\Omega t}{2} \\ -i \sin \frac{\Omega t}{2} & \cos \frac{\Omega t}{2} \end{pmatrix}, \quad |\psi(0)\rangle = |0\rangle = (1, 0)^T.$$

(a)

$$|\psi(t)\rangle = U(t)|0\rangle = \begin{pmatrix} \cos \frac{\Omega t}{2} \\ -i \sin \frac{\Omega t}{2} \end{pmatrix}.$$

Populations:

$$p_0(t) = \cos^2(\Omega t/2) = \frac{1}{2}(1 + \cos \Omega t), \quad p_1(t) = \sin^2(\Omega t/2) = \frac{1}{2}(1 - \cos \Omega t).$$

Check: $p_0 + p_1 = 1$ ✓; $p_0(0) = 1$, $p_1(\pi/\Omega) = 1$ (full inversion at the π -pulse time).

(b) **Sketch.** $p_0(t)$ is a cosine oscillating between 1 and 0 with period $2\pi/\Omega$: $p_0 = 1$ at $t = 0, 2\pi/\Omega, 4\pi/\Omega, \dots$ and $p_0 = 0$ at $t = \pi/\Omega, 3\pi/\Omega, \dots$ — classic Rabi flopping.

Q3 — LC oscillator wave functions

Given $\hat{\phi} = \Phi_{\text{ZPF}}(\hat{a} + \hat{a}^\dagger)$ and the ground state

$$\Psi_0(\phi) = \frac{1}{(2\pi\Phi_{\text{ZPF}}^2)^{1/4}} \exp\left(-\frac{\phi^2}{4\Phi_{\text{ZPF}}^2}\right).$$

Use $\hat{a}^\dagger = \hat{\phi}/\Phi_{\text{ZPF}} - \hat{a}$ together with $\hat{a}\Psi_0 = 0$.

(a) **One-photon state:**

$$\Psi_1(\phi) = \hat{a}^\dagger \Psi_0(\phi) = \frac{\phi}{\Phi_{\text{ZPF}}} \Psi_0(\phi).$$

Two-photon state:

$$\Psi_2(\phi) = \frac{1}{\sqrt{2}} \hat{a}^\dagger \Psi_1 = \frac{1}{\sqrt{2}} \left(\frac{\phi}{\Phi_{\text{ZPF}}} \Psi_1 - \hat{a} \Psi_1 \right).$$

Since $\hat{a} \Psi_1 = \hat{a} \hat{a}^\dagger \Psi_0 = (1 + \hat{a}^\dagger \hat{a}) \Psi_0 = \Psi_0$,

$$\Psi_2(\phi) = \frac{1}{\sqrt{2}} \left[\left(\frac{\phi}{\Phi_{\text{ZPF}}} \right)^2 - 1 \right] \Psi_0(\phi).$$

(If the problem uses the unnormalized raising — i.e. $\Psi_{n+1} = \hat{a}^\dagger \Psi_n$ without the $1/\sqrt{n+1}$ factor — then drop the $1/\sqrt{2}$ and recover the form in the problem statement.)

(b) Superposition $\Psi_+ = \frac{1}{\sqrt{2}}(\Psi_0 + \Psi_1) = \frac{1}{\sqrt{2}}(1 + \phi/\Phi_{\text{ZPF}})\Psi_0(\phi)$.

Shape: a Gaussian centered at $\phi = 0$ multiplied by the linear factor $(1 + \phi/\Phi_{\text{ZPF}})$. The Gaussian envelope's peak shifts toward $\phi > 0$ and the wave function vanishes at $\phi = -\Phi_{\text{ZPF}}$; for $\phi \gg \Phi_{\text{ZPF}}$ it falls as $\phi e^{-\phi^2/4\Phi_{\text{ZPF}}^2}$. Plot qualitatively:

- Zero at $\phi = -\Phi_{\text{ZPF}}$.
- Negative lobe for $\phi < -\Phi_{\text{ZPF}}$.
- Maximum near $\phi \approx +\Phi_{\text{ZPF}}$.
- Gaussian decay for $|\phi| \gtrsim 2\Phi_{\text{ZPF}}$.

This is the classic “displaced vacuum” intuition — superposition of $|0\rangle$ and $|1\rangle$ has a non-zero $\langle\phi\rangle$ (in contrast to any pure Fock state).