

Quantum Hardware

Quantum Measurements

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Term 252

Introduction to Experimental Quantum Measurements

Lecture Notes - 03

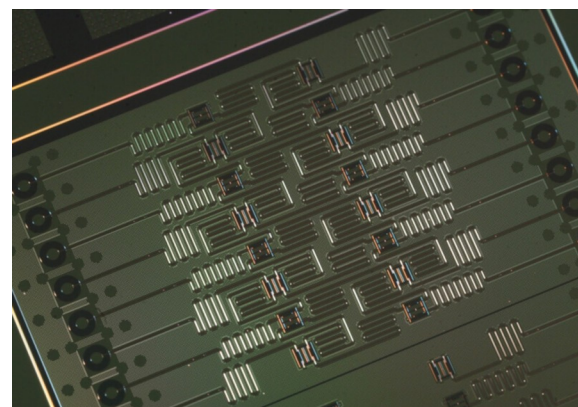
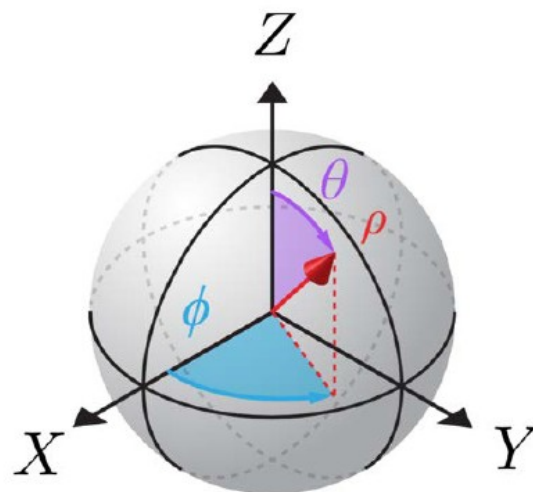
Qubit Ideally and in Reality

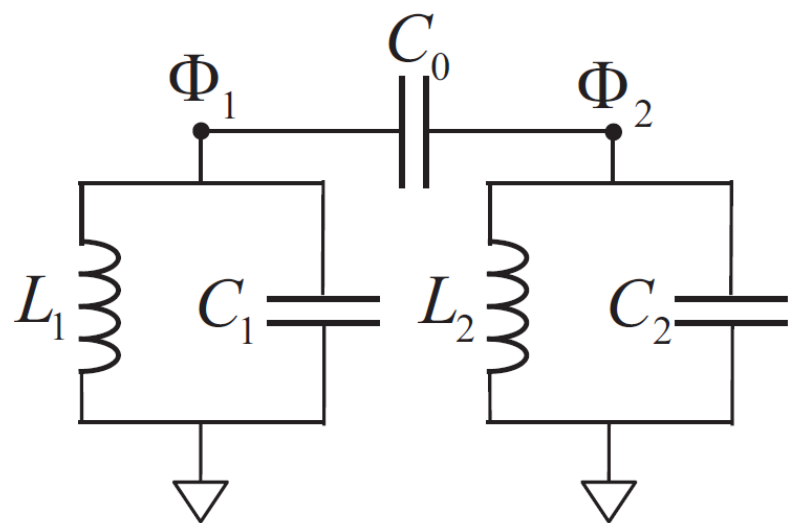
Energy levels

— $|1\rangle$

— $|0\rangle$

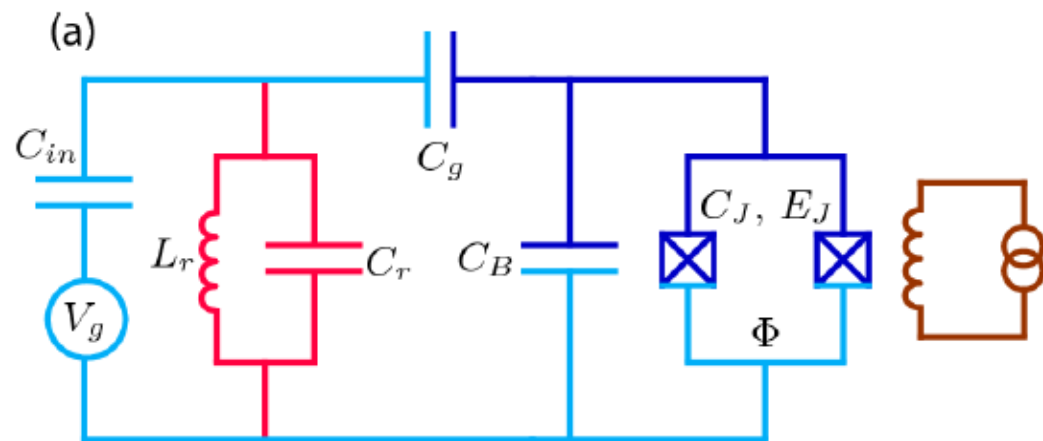
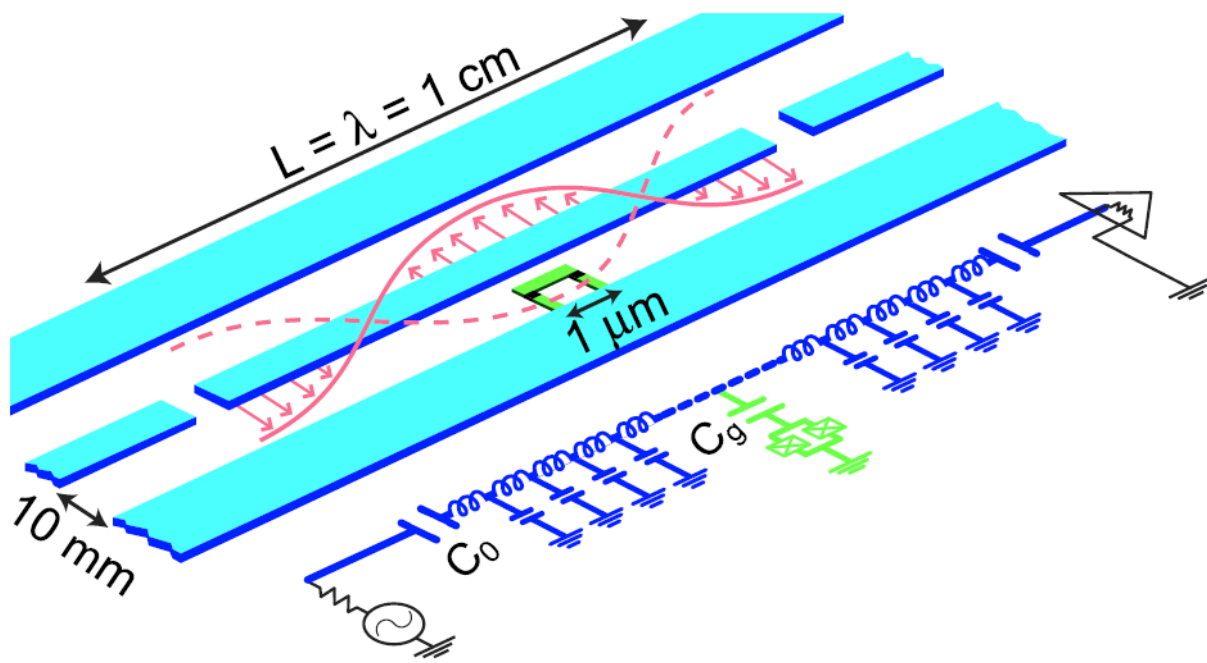
Hilbert space



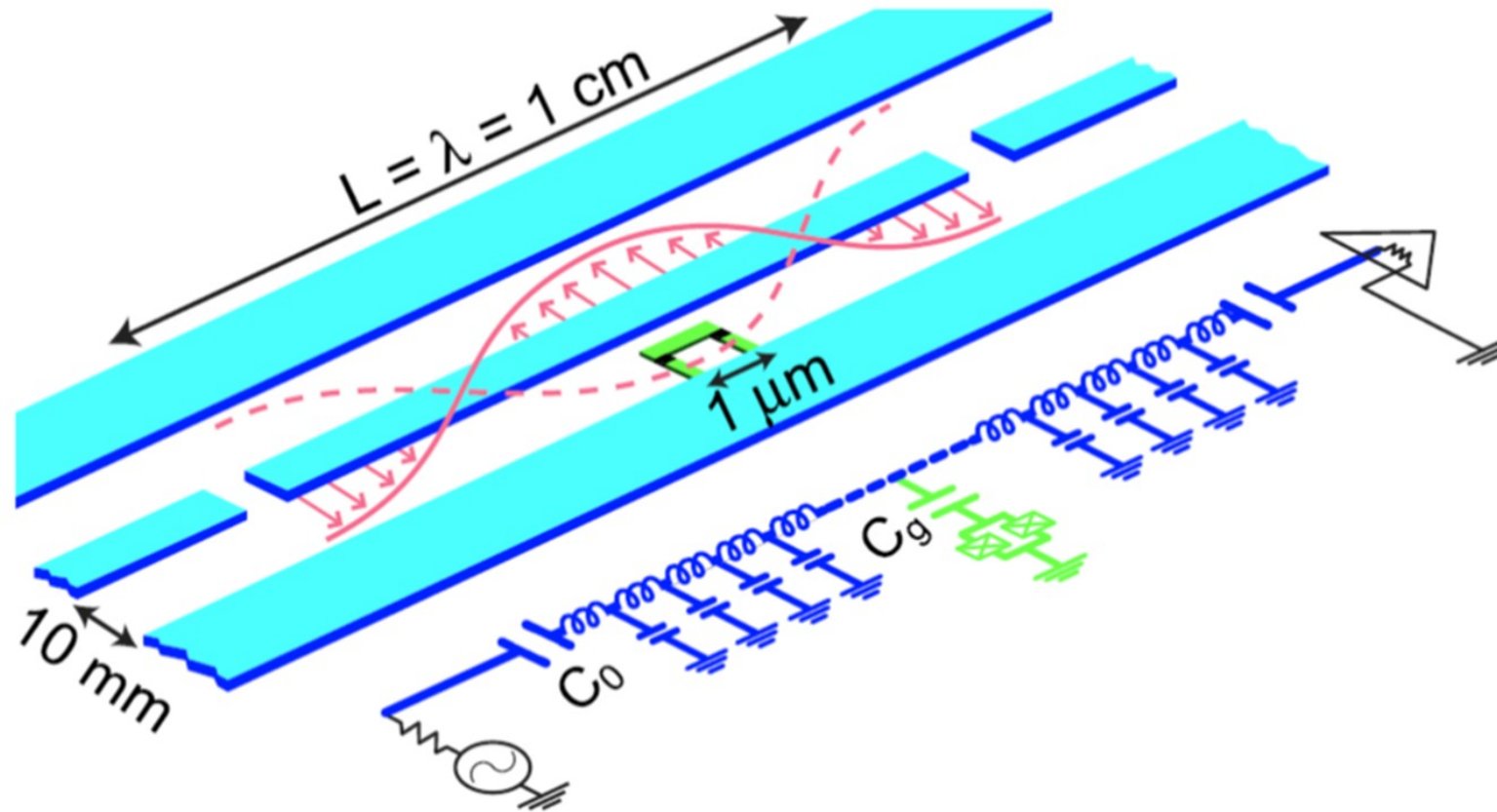


Choosing the fluxes Φ_1 and Φ_2 as the coordinates of the two oscillators, the Lagrangian can be written

$$\mathcal{L} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_0[\dot{\Phi}_1 - \dot{\Phi}_2]^2 - \frac{1}{2L_1}\Phi_1^2 - \frac{1}{2L_2}\Phi_2^2 \quad (2.110)$$



Hamiltonian of the transmission line (using Lumped elements)



L – inductance per unit length
 C – Capacitance per unit length

Coherent State of Harmonic Oscillator

$$\hat{H} = \frac{\hbar\Omega}{2} [\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger] = \hbar\Omega [\hat{a}^\dagger \hat{a} + 1/2] \quad \text{Hamiltonian of a harmonic oscillator}$$

Raising lowering operators

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

Where n is Fock state of the harmonic oscillator

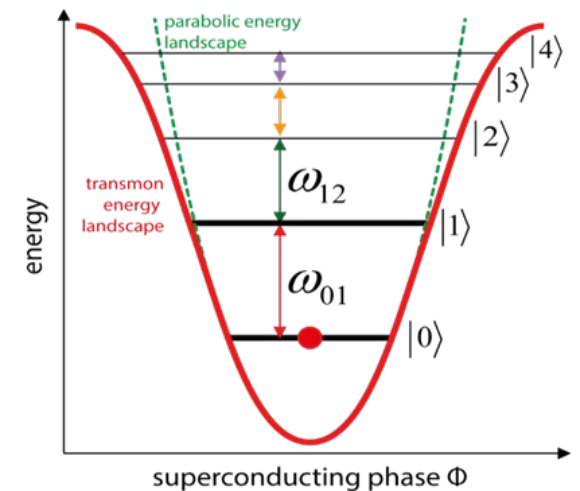
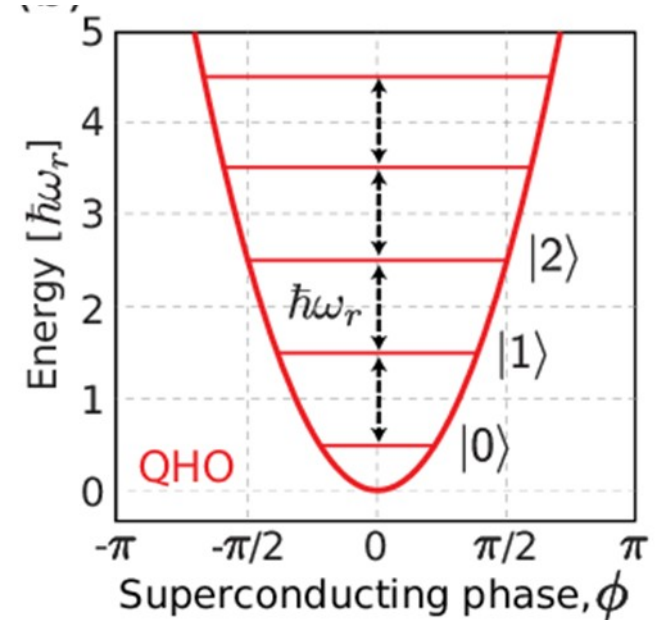
$$\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$$

$$\hat{a}^\dagger \hat{a} = \hat{n} \text{ called number operator}$$

$$\hat{n} |n\rangle = n |n\rangle$$

Coherent state $|\alpha\rangle$, is a special state of a resonator, which is very useful in **quantum optics and circuit QED**, because it behaves almost classically. It can be defined to be the eigenstate of an annihilation operator

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$$



Harmonic to Anharmonic Oscillator

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \hat{\phi}^2$$

$$\omega_{LC} = \omega = \frac{1}{\sqrt{LC}}$$

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega^2 \hat{\phi}^2$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$



To diagonalize the Hamiltonian we introduced and expressed the $\hat{Q}$, $\hat{\phi}$, and $\hat{H}$ in-terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2)$$

Now the let us write Hamiltonian of oscillator with JJ

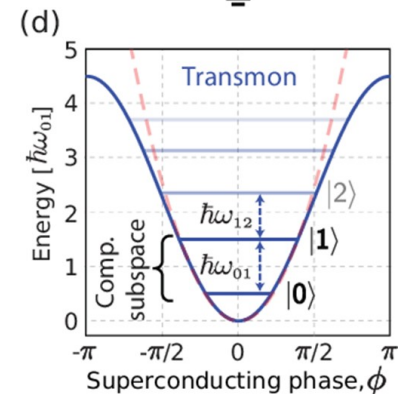
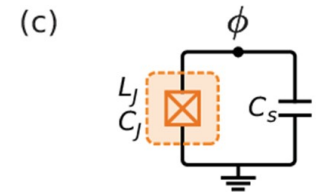
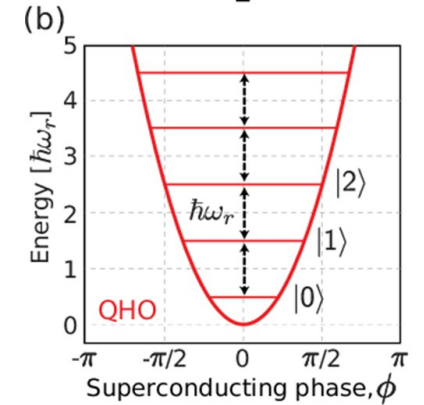
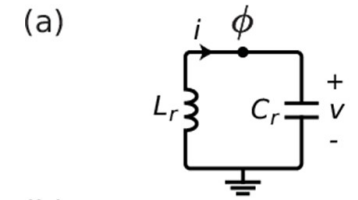
$$\hat{H} = \frac{\hat{Q}^2}{2C} + E_J (1 - \cos(\Delta\hat{\phi}))$$

Q (charge) can be replaces by $2ne$
Because in SC current is because of cooper-pairs

$$\hat{H} = \frac{(2e)^2}{2C} \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi}))$$

Instead of $\hat{Q}$ now we have $\hat{n}$ (charge number operator)

$$\hat{H} = 4 \frac{e^2}{2C} \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi})) \quad \longrightarrow \quad \hat{H} = 4E_C \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi}))$$



Harmonic to Anharmonic Oscillator

Hamiltonian of oscillator with JJ

$$\hat{H} = 4E_c \hat{n}^2 + E_J (1 - \cos(\Delta \hat{\phi}))$$

Using cosine series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\hat{H} = 4E_c \hat{n}^2 + E_J \left\{ \frac{(\Delta \hat{\phi})^2}{2!} - \frac{(\Delta \hat{\phi})^4}{4!} + \frac{(\Delta \hat{\phi})^6}{6!} - \frac{(\Delta \hat{\phi})^8}{8!} + \dots \right\}$$

$$\hat{H} = \underbrace{4E_c \hat{n}^2 + \frac{E_J}{2} (\Delta \hat{\phi})^2}_{\text{LHO}} - \underbrace{\frac{E_J}{24} (\Delta \hat{\phi})^4 - \dots}_{\text{Non-linear effects are from these terms}}$$

LHO

Non-linear effects are from these terms

Using convention process we can express above Hamiltonian in terms of ladder operators $\hat{b}$ and $\hat{b}^\dagger$ (substituting $\hbar = 1$)

$$\hat{H} = \omega_J \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) - \frac{E_c}{12} (\hat{b}^\dagger + \hat{b})^4$$

Comparing this Hamiltonian with that of LHO

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \hat{\phi}^2$$



$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_{LC}^2 \hat{\phi}^2$$

$$4E_c \leftrightarrow \frac{1}{2C}$$

$$\frac{E_J}{2} \leftrightarrow \frac{1}{2L}$$

$$\omega_J = \sqrt{8E_J E_c} \leftrightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}$$

$$\hat{H} = \omega_J \hat{b}^\dagger \hat{b} - \frac{E_c}{12} (\hat{b}^\dagger + \hat{b})^4$$

$$\hat{H} = \omega_J \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + 2 \hat{b}^\dagger \hat{b})$$

$$\hat{H} = (\omega_J - E_c) \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})$$

$$\hat{H} = \omega_{01} \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})$$

$$\hat{H} = -\frac{1}{2} \omega_{01} \sigma_z$$

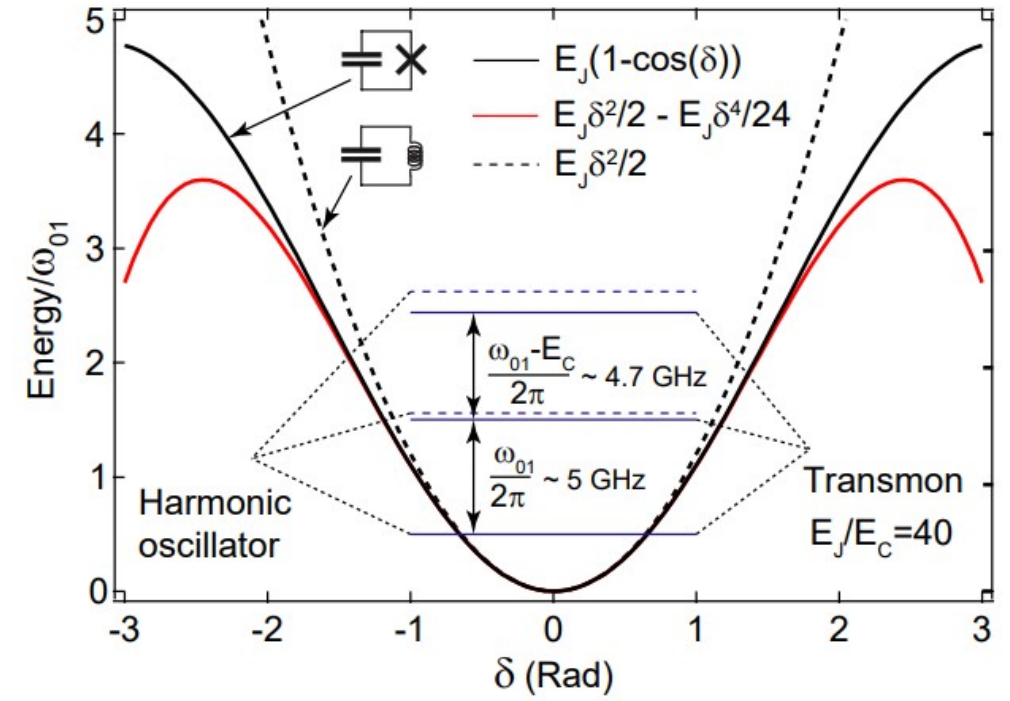


Figure 2.9: **Transmon energy levels:** A typical transmon ($E_J/E_C = 40$) potential (Eq (2.2.8) in solid black curve) in comparison with a nonlinear oscillator (Eq. (2.2.9) in the solid red curve) and a harmonic oscillator (dashed parabola). The first three energy levels are also depicted for the transmon (nonlinear oscillator) in comparison to the harmonic oscillator. The typical values for transition energies/frequencies are shown. Note $E_{01} = \hbar\omega_{01}$ and we set $\hbar = 1$.

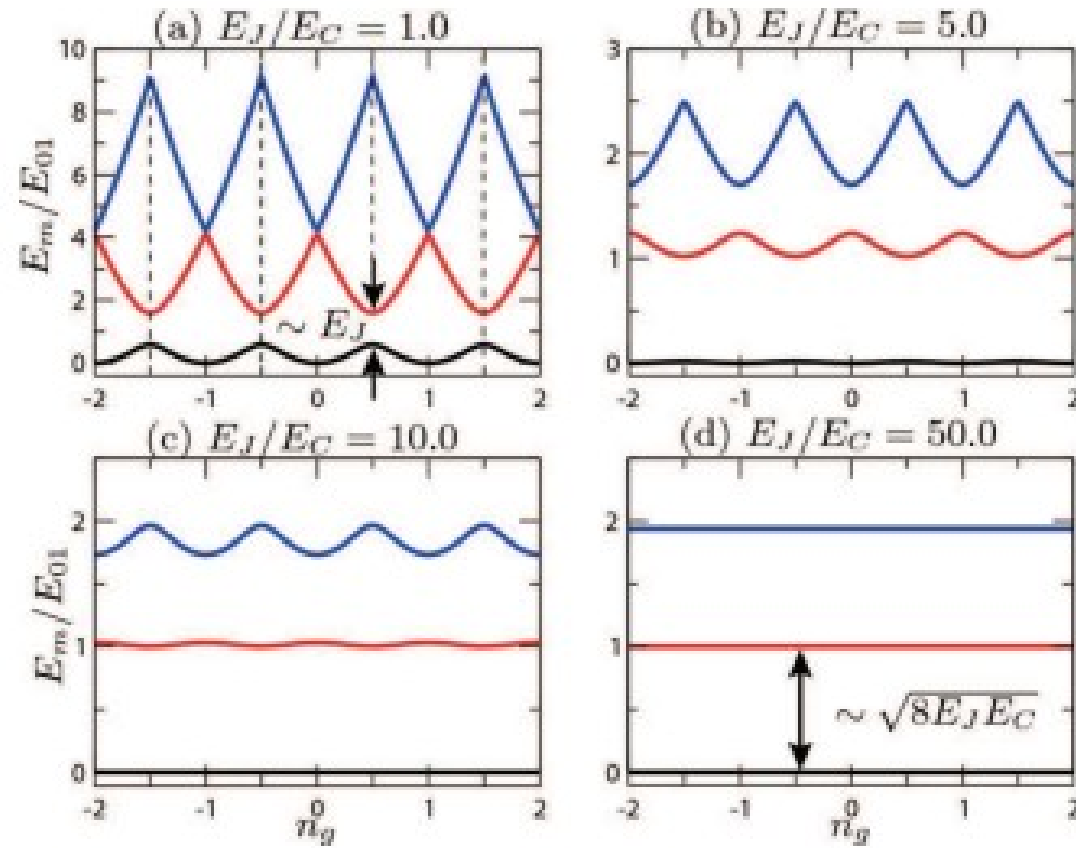


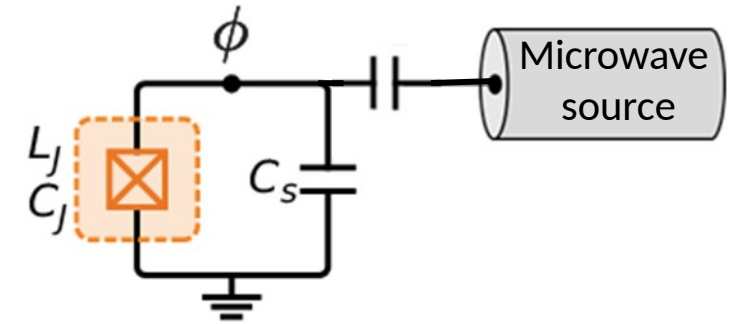
Fig. 4.6 Energy spectrum of the Cooper pair box as a function of offset charge for different values of the dimensionless ratio of Josephson energy to charging energy. The exponential decrease in the charge dispersion is clearly seen. From (Koch *et al.*, 2007).

Qubit Coupled to a Source

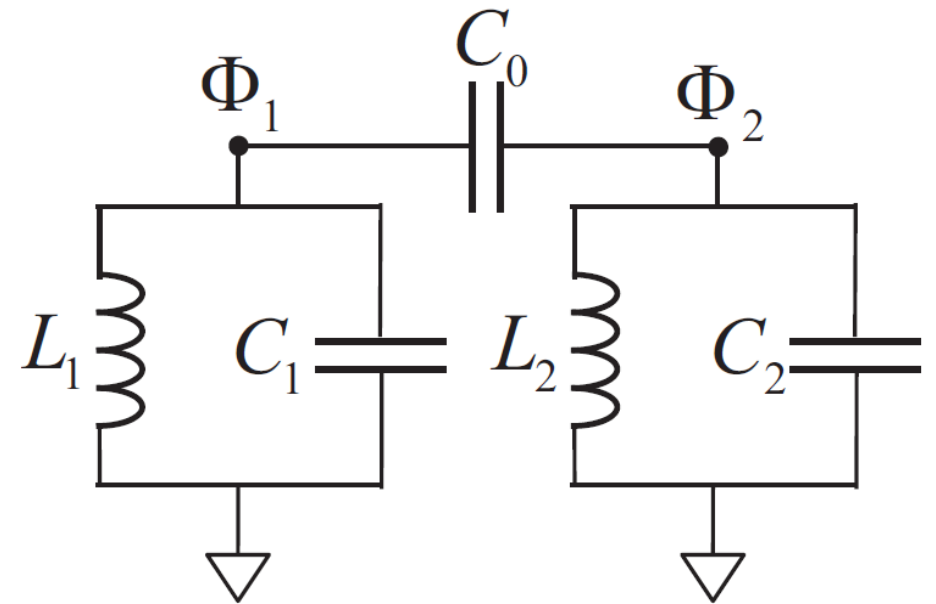
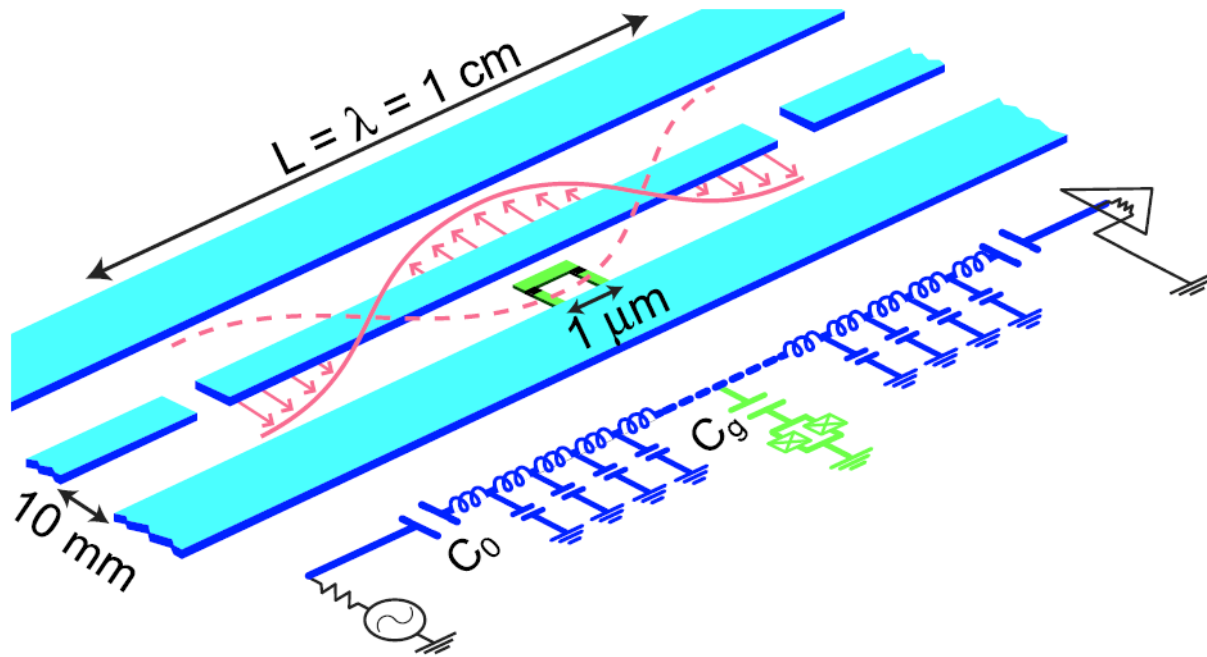
You can change the state of qubit by applying suitable MW source

$$\hat{H} = 4E_c \hat{n}^2 + E_J(1 - \cos(\Delta\hat{\phi}))$$

$$\hat{H} = 4E_c (\hat{n} - n_g)^2 + E_J(1 - \cos(\Delta\hat{\phi}))$$



Qubit Cavity Interaction



Qubit-Cavity Coupled States

$$H_{int} = -\hat{d} \cdot \hat{E}_x\left(\frac{L}{2}, t\right), \text{ where } \hat{d} = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}. \quad (2.3.1)$$

$$\hat{H}_I = \hbar g (\hat{\sigma} + \hat{\sigma}^\dagger)(\hat{a} + \hat{a}^\dagger)$$

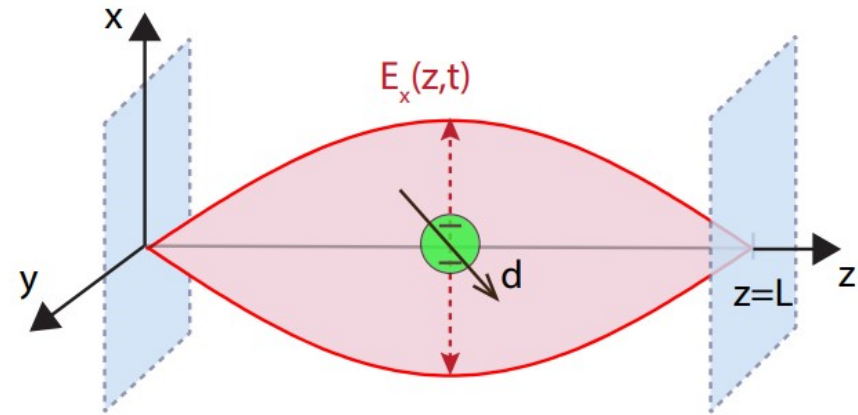
$$\hat{H}_I = \hbar g (\hat{\sigma} \hat{a} + \hat{\sigma} \hat{a}^\dagger + \hat{\sigma}^\dagger \hat{a} + \hat{\sigma}^\dagger \hat{a}^\dagger)$$

Under RWA

$$\hat{H}_I = \hbar g (\hat{\sigma} \hat{a}^\dagger + \hat{\sigma}^\dagger \hat{a})$$

$$\hat{H}_{JC} = \hbar \omega \hat{\sigma}^\dagger \hat{\sigma} + \hbar \omega [\hat{a}^\dagger \hat{a} + 1/2] + \hbar g (\hat{\sigma} \hat{a}^\dagger + \hat{\sigma}^\dagger \hat{a})$$

$$\hat{H}_{JC} = -\frac{\hbar \omega}{2} \hat{\sigma}_z + \hbar \omega [\hat{a}^\dagger \hat{a} + 1/2] + \hbar g (\hat{\sigma} \hat{a}^\dagger + \hat{\sigma}^\dagger \hat{a})$$



$$H_{int} = -g(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-),$$

$$H_{JC} = \omega_c(\hat{a}^\dagger \hat{a} + \frac{1}{2}) - \frac{1}{2} \omega_q \sigma_z - g(\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+).$$

$|g\rangle|0\rangle \rightarrow$ qubit in ground state, no photons in the cavity

$|g\rangle|n+1\rangle \rightarrow$ qubit in ground state, $n+1$ photons in the cavity

$|e\rangle|n\rangle \rightarrow$ qubit in excited state, n photons in the cavity

$$H_{JC} = \begin{pmatrix} \frac{1}{2}\omega_c - \frac{\omega_g}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\omega_c - \frac{\omega_g}{2} & g & 0 & 0 & 0 \\ 0 & g & \frac{3}{2}\omega_c + \frac{\omega_g}{2} & 0 & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & 0 & (n + \frac{1}{2})\omega_c - \frac{\omega_g}{2} & \sqrt{n+1}g \\ 0 & 0 & 0 & 0 & \sqrt{n+1}g & (n + \frac{1}{2})\omega_c + \frac{\omega_g}{2} \end{pmatrix}, \quad (2.3.9)$$

which shows the Hamiltonian is block-diagonal and all blocks follow a general form (except the first block which has only one element $\frac{1}{2}\omega_c - \frac{\omega_g}{2}$ corresponding to the absolute ground state of the system). Having a block-diagonal Hamiltonian makes it easy to find its eigenvalues. We only need to diagonalized individual blocks and the resulting eigenvalues of each block indeed are the eigenvalues of the entire Hamiltonian. For each block M_n we have,

$$M_n = \begin{pmatrix} (n + \frac{1}{2})\omega_c - \frac{\omega_g}{2} & \sqrt{n+1}g \\ \sqrt{n+1}g & (n + \frac{1}{2})\omega_c + \frac{\omega_g}{2} \end{pmatrix}, \quad (2.3.10)$$

where $n = 1, 2, \dots$. The eigenstates of M_n and $|g\rangle|0\rangle$ corresponding to $n = 0$, form a complete set of eigenstates for the entire qubit-cavity system. For the eigenvalues we have,

$$E_g = -\frac{\Delta}{2} \quad (2.3.11)$$

$$E_{\mp} = (n+1)\omega_c \mp \frac{1}{2}\sqrt{4g^2(n+1) + \Delta^2}. \quad (2.3.12)$$

$$E_g = -\frac{\Delta}{2} \quad (2.3.11)$$

$$E_{\mp} = (n+1)\omega_c \mp \frac{1}{2}\sqrt{4g^2(n+1) + \Delta^2}. \quad (2.3.12)$$

where $\Delta = \omega_q - \omega_c$. The eigenstate associated with each of these eigenvalues are called the *dressed states* of the qubit and cavity,

$$|0, -\rangle = |g\rangle|0\rangle \quad (2.3.13)$$

$$|n, -\rangle = \cos(\theta_n)|g\rangle|n+1\rangle - \sin(\theta_n)|e\rangle|n\rangle \quad (2.3.14)$$

$$|n, +\rangle = \sin(\theta_n)|g\rangle|n+1\rangle + \cos(\theta_n)|e\rangle|n\rangle \quad (2.3.15)$$

where $\theta_n = \frac{1}{2} \tan^{-1}(2g\sqrt{n+1}/\Delta)$ which quantifies the “level of hybridization”. In the limit of $\Delta \rightarrow 0$ where qubit and cavity have the same energy we have $\theta_n = \pi/4$ and the dressed states are in maximum hybridization,

$$|n, \mp\rangle = \frac{1}{\sqrt{2}} (|g\rangle|n+1\rangle \mp |e\rangle|n\rangle), \quad (2.3.16)$$

which means each of the dressed states has a 50 %-50 % characteristic of the cavity photon and qubit excitations. These states are called *polaritons*. The energy difference between the first two polariton states is $2g$.

A nice way to look at dressed state energy levels is by comparing them to the corresponding uncoupled system energy levels, the bare states. For that, consider Figure 2.11 where we display the energy levels of an uncoupled qubit-cavity system compared to the dressed state energy levels for different values of the qubit-cavity detuning. The bare state energy levels are depicted by solid black lines. The dressed states are depicted by bars that are color-coded by blue (red) for cavity- (qubit-) like states. In the first panel, the qubit and cavity are far detuned ($\Delta \ll 0$) which means $\theta_n \simeq 0$ and the effective coupling is negligible. Therefore the dressed states energy levels almost overlap with the uncoupled cavity-qubit state, the bare states (as depicted in panel 1). In the second panel, we change the energy level for the qubit. The detuning Δ is still negative but it is getting smaller and smaller in terms of magnitude. The dressed states start pushing away each other and deviate from the corresponding bare states. In this situation, $\theta_n \in (0, \pi/4)$ and the upper dressed state acquires some qubit character, and similarly, the lower dressed state acquire some photon character. In panel three $\Delta = 0$ and the hybridization is its maximum level, $\theta_n = \pi/4$ and the dressed states (which we now call polaritons) push each other away and deviate maximally from the bare states. The separation between two polaritons is $2g$. Now both

Qubit-Cavity Coupled States

$$|n, -\rangle = \cos(\theta_n)|0\rangle|n+1\rangle - \sin(\theta_n)|1\rangle|n\rangle$$

$$|n, +\rangle = \sin(\theta_n)|0\rangle|n+1\rangle + \cos(\theta_n)|1\rangle|n\rangle$$

$$\theta_n = \frac{1}{2} \tan^{-1} [2g\sqrt{n+1}/\Delta]$$

$$\Delta = \omega_q - \omega_c$$

Detuning

$$E_{\pm} = (n+1)\omega_c \pm \frac{1}{2} \sqrt{4g^2(n+1) + \Delta^2}$$

$$|n, -\rangle = \cos(\theta/2)|0\rangle|n+1\rangle - \sin(\theta/2)|1\rangle|n\rangle$$

$$|n, +\rangle = \sin(\theta/2)|0\rangle|n+1\rangle + \cos(\theta/2)|1\rangle|n\rangle$$

$$\tan\theta = -\frac{\Omega_n}{\Delta}$$

$$\Omega_n = 2g\sqrt{n+1}$$

Rabi frequency

$$\Omega_n^{\Delta} = \sqrt{\Omega_n^2 + \Delta^2}$$

Detuned system Rabi frequency

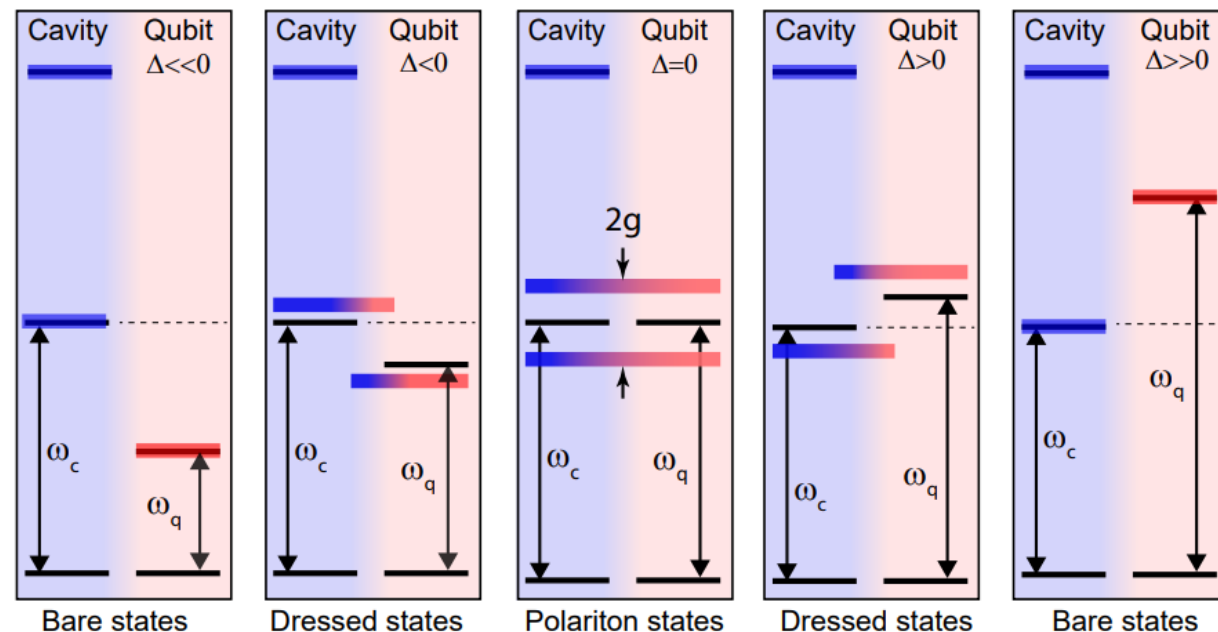


Figure 2.11: **Dressed states vs bare states:** The panels illustrate the dressed states of the qubit-cavity system for different qubit-cavity detunings in comparison with the bare states (refer to the main text for a more detailed description). Note that this illustration is not accurate and lacks some details but we rather to avoid them here.

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{\Omega_n^{\Delta} - \Delta}{2\Omega_n^{\Delta}}}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{\Omega_n^{\Delta} + \Delta}{2\Omega_n^{\Delta}}}$$

Qubit-Cavity Coupled States

In the absence of detuning

$$|\psi_0\rangle = |0\rangle|n+1\rangle = |0, n+1\rangle$$

$$|\psi_t\rangle = \cos(g\sqrt{n+1}t)|0, n+1\rangle - i\sin(g\sqrt{n+1}t)|1, n\rangle$$

$$p_{|0\rangle} = p_{0,n+1} = \cos^2(g\sqrt{n+1}t)$$

$$p_{|0\rangle} = p_{0,n+1} = \frac{1}{2} \left(1 + \cos(2g\sqrt{n+1}t) \right)$$

$$p_{|0\rangle} = p_{0,n+1} = \frac{1}{2} \left(1 + \cos(\Omega_n t) \right)$$

$$\Omega_n = 2g\sqrt{n+1}$$

Rabi frequency

Qubit-Cavity Coupled States

$$p_{|0\rangle} = p_{0,n+1} = \cos^2(g\sqrt{n+1}t)$$

$$p_{|0\rangle} = p_{0,n+1} = \frac{1}{2}(1 + \cos(\Omega_n t))$$

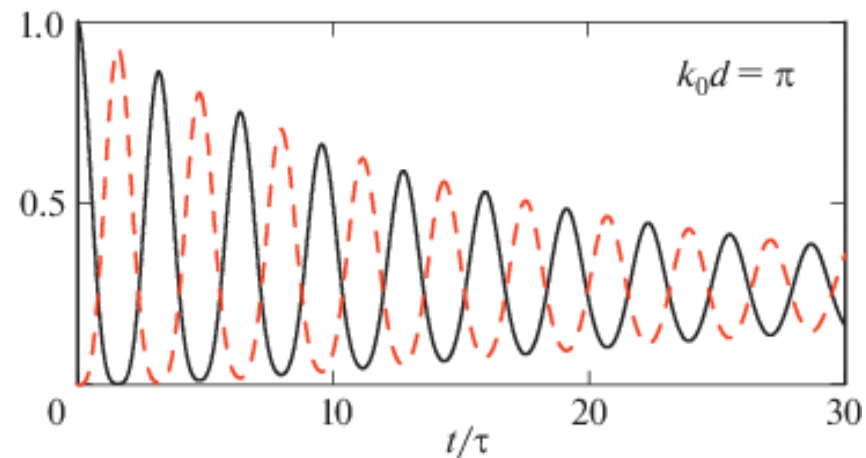
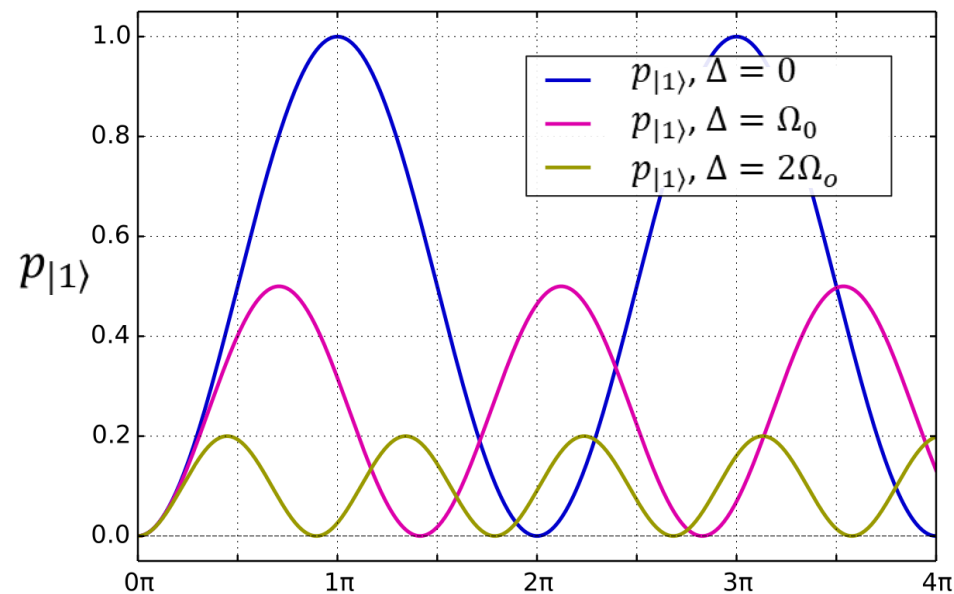
$$\Omega_n = 2g\sqrt{n+1}$$

Rabi frequency depends upon coupling strength and number of photons in cavity

During computing we operate quantum chip at $n \approx 1$

Detuning

$$\Delta = \omega_q - \omega_c$$



The first way is to solve the Schrödinger equation for this time-dependent Hamiltonian. We start with an *ansatz* instead of starting from scratch. The idea is that if we have no electric field or turn off the interaction, then we know the solution for Hamiltonian (2.4.2) would be $|\psi\rangle = C_g|g\rangle + C_e|e\rangle$ and its time evolution would be $|\psi(t)\rangle = C_g e^{+i\frac{\omega_q}{2}t}|g\rangle + C_e e^{-i\frac{\omega_q}{2}t}|e\rangle$. Now, we hope to find the solutions for (2.4.2) in the form of,

$$|\psi(t)\rangle = C_g(t)e^{+i\frac{\omega_q}{2}t}|g\rangle + C_e(t)e^{-i\frac{\omega_q}{2}t}|e\rangle, \quad (2.4.3)$$

where we just let the coefficients also be time-dependent. Now we plug this ansatz into the Schrödinger equation,

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \left(-\frac{1}{2}\omega_q\sigma_z - A\cos(\omega_d t)\sigma_x\right)|\psi(t)\rangle \quad (2.4.4)$$

By substituting Equation (2.4.3) into (2.4.4), one can obtain two coupled ordinary differential equations (ODEs) for $C_g(t)$ and $C_e(t)$,

$$\dot{C}_g = iA\cos(\omega_d t)e^{-i\omega_q t}C_e, \quad (2.4.5a)$$

$$\dot{C}_e = iA\cos(\omega_d t)e^{+i\omega_q t}C_g. \quad (2.4.5b)$$

$$\dot{C}_g = iA e^{-i(\omega_q - \omega_d)t} C_e, \quad (2.4.6a)$$

$$\dot{C}_e = iA e^{+i(\omega_q - \omega_d)t} C_g. \quad (2.4.6b)$$

$$C_g(t) = \frac{e^{-i\frac{\Delta_d}{2}t}}{\Omega_R} \left(\Omega_R \cos\left(\frac{\Omega_R}{2}t\right) + i\Delta_d \sin\left(\frac{\Omega_R}{2}t\right) \right) \quad (2.4.7a)$$

$$C_e(t) = i \frac{A e^{+i\frac{\Delta_d}{2}t}}{\Omega_R} \sin\left(\frac{\Omega_R}{2}t\right), \quad (2.4.7b)$$

where $\Delta_d = \omega_q - \omega_d$ and $\Omega_R = \sqrt{A^2 + \Delta_d^2}$. Having the solution for $|\psi(t)\rangle$, we can obtain the evolution for any relevant observables. In order to see what the dynamics look like, we may look at the population of the qubit excited state,

$$P_e(t) = |C_e(t)|^2 = \frac{A^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R}{2}t\right). \quad (2.4.8)$$

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As depicted in Figure 2.13, the qubit doesn't respond that much to a far detuned drive, but as the detuning gets smaller the oscillations grow. For an on-resonant drive ($\Delta_d = 0$), we have slowest but highest contrast oscillations of the qubit populations.

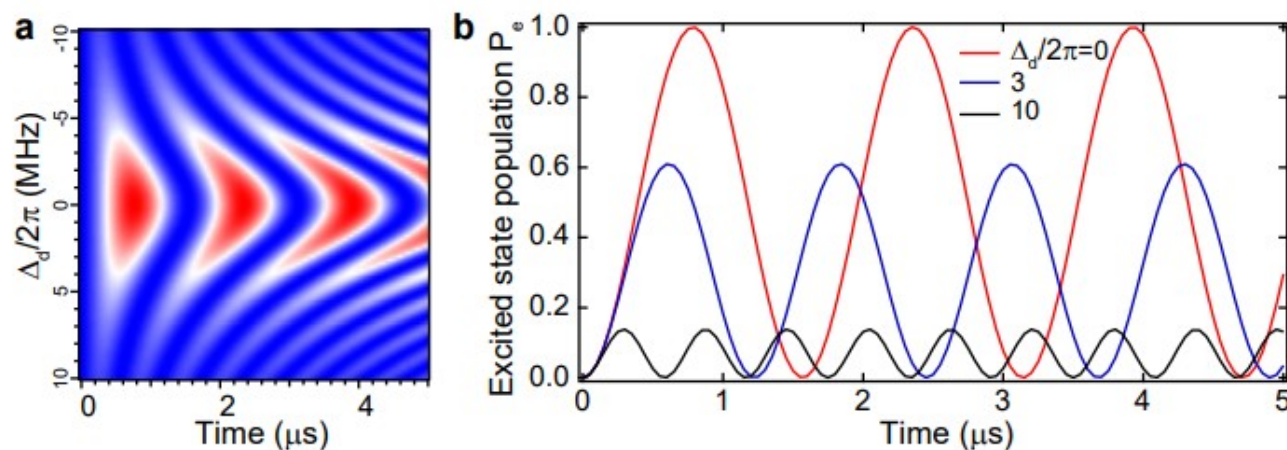
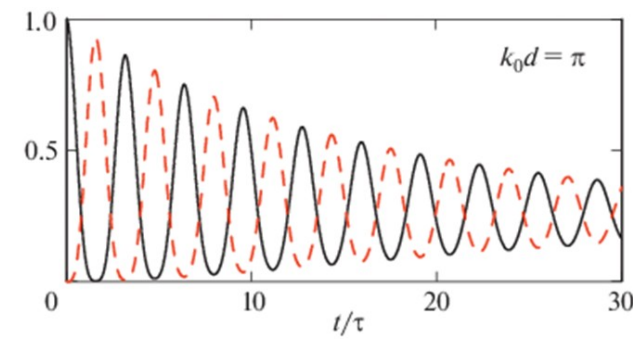


Figure 2.13: **Rabi oscillations:** **a**, The chevron plot. The excited state population P_e versus time for different detunings Δ_q . **b**, Three cuts from the chevron plot at different detuning values. The on-resonant drive gives the maximum contrast for the oscillations.



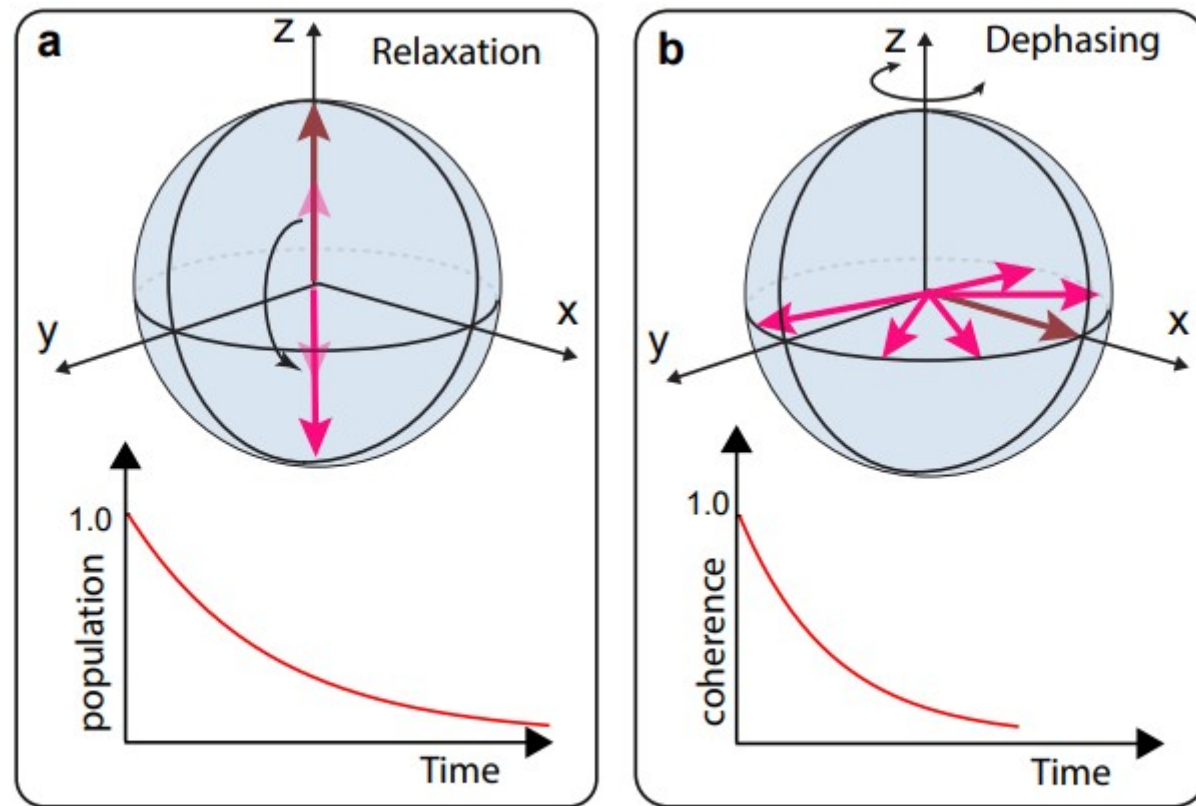


Figure 2.16: Relaxation and dephasing of a qubit.

Dispersive Approximation

$$\Delta \gg g$$

JC – Jaynes Cumming Hamiltonian in the dispersive limit

$$\hat{H}_{\text{dis}} = \omega_c \hat{a}^\dagger \hat{a} - \frac{1}{2} \omega_q \sigma_z - \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \sigma_z. \quad (2.3.19)$$

The dispersive Hamiltonian (2.3.19) describes the situation where the cavity and qubit are far detuned and coupling is weak and dressed states are almost overlapping with the bare states (see Figure 2.11 panel 1). Yet, there is a very small interaction as described by the last term in Equation (2.3.19). In order to make better sense of this interaction, we re-arrange the terms in Equation (2.3.19) as follows,

$$\hat{H}_{\text{dis}} = (\omega_c - \chi \sigma_z) \hat{a}^\dagger \hat{a} - \frac{1}{2} \omega_q \sigma_z, \quad (2.3.20)$$

where $\chi = \frac{g^2}{\Delta}$ is the dispersive shift or dispersive coupling rate³. We see that the dispersive interaction is manifested as a qubit-state-dependent frequency shift for the cavity. If the qubit is in the ground (excited) state $|g\rangle$ ($|e\rangle$) then $\langle \sigma_z \rangle = 1$ ($\langle \sigma_z = -1 \rangle$) which means that the cavity frequency shifts by $+\chi$ ($-\chi$). Therefore one can detect this frequency shift for the cavity to determine the state of the qubit.

Avoided Crossing

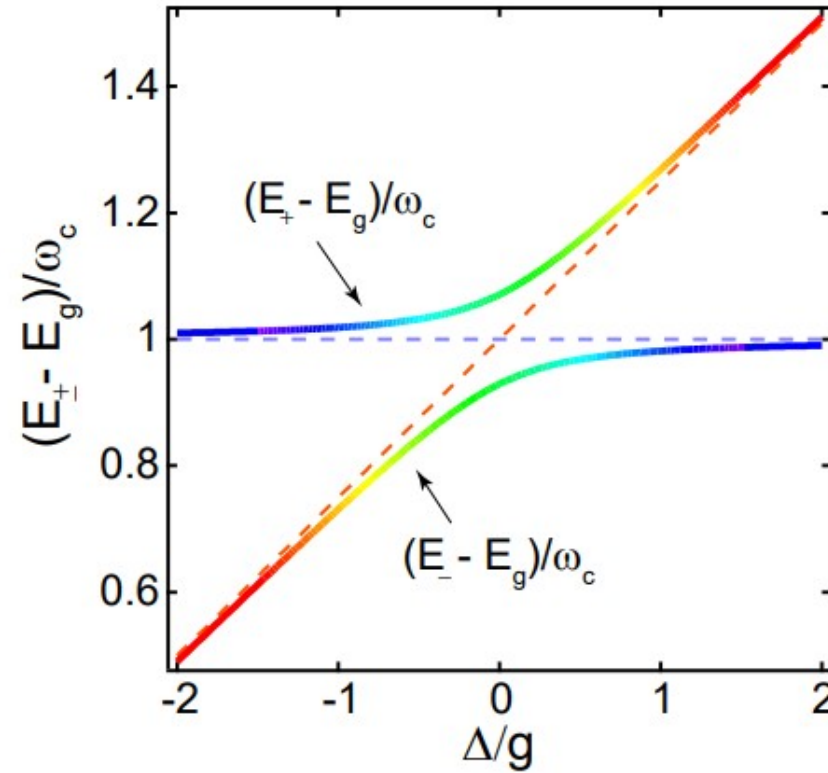
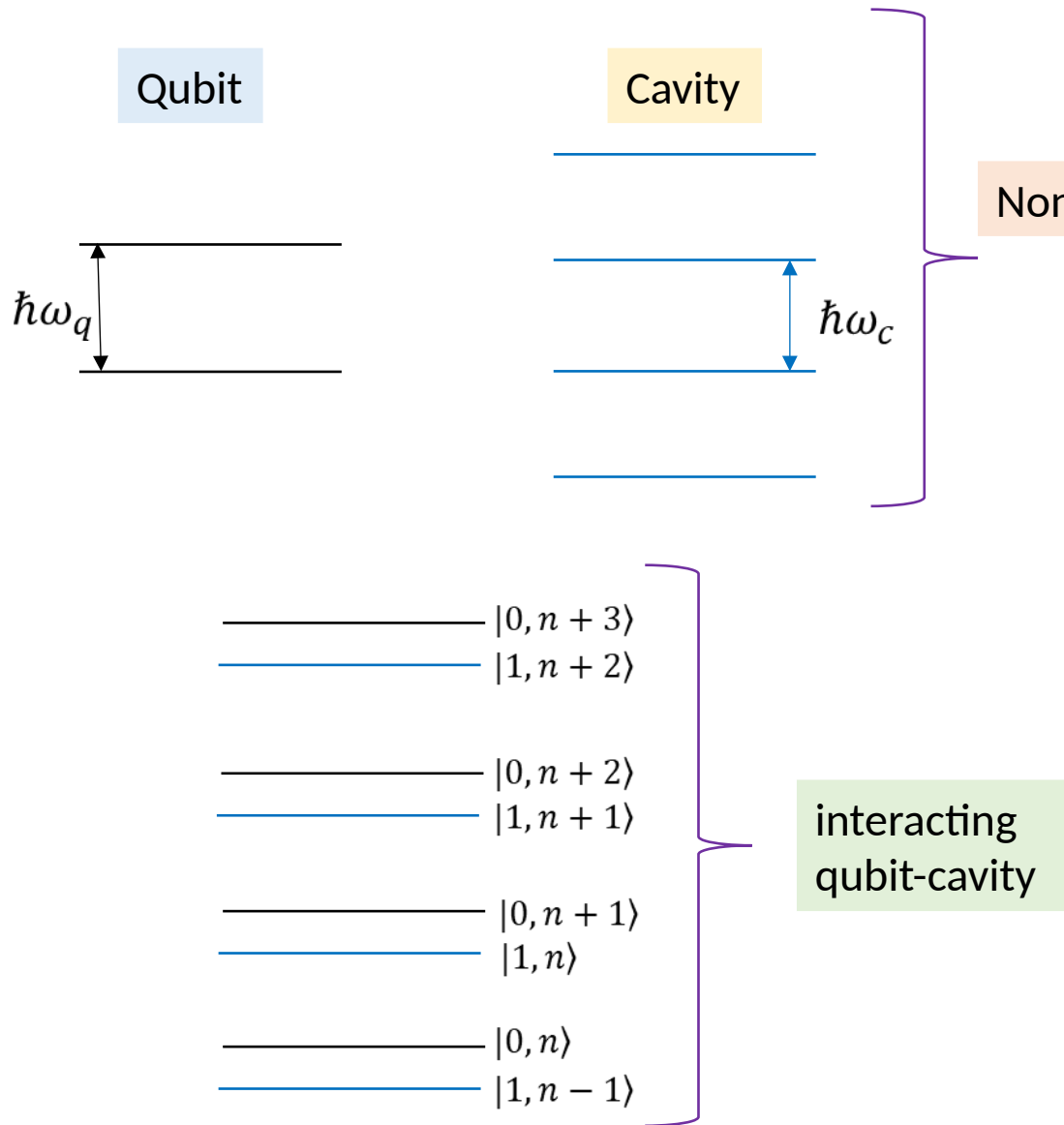


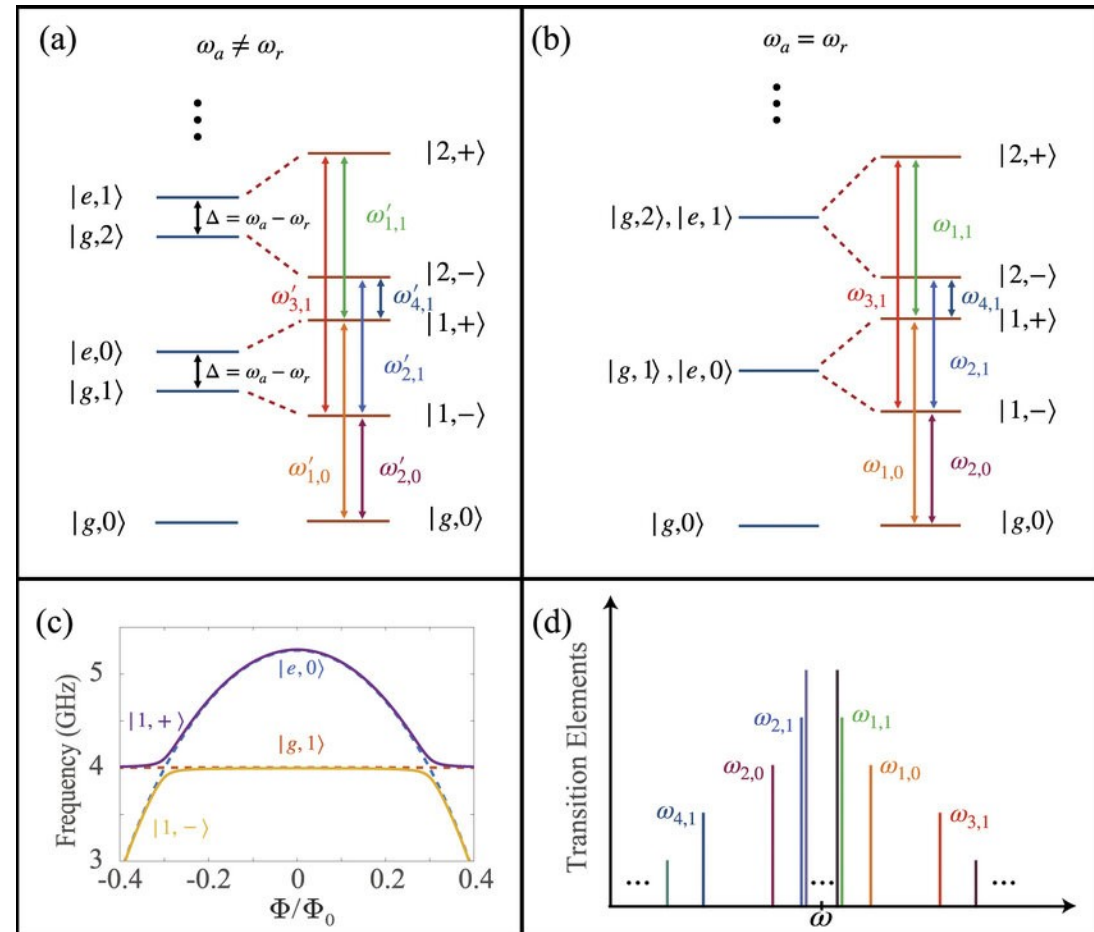
Figure 2.12: **Avoided crossing:** The transition energy from higher and lower dressed states to the ground state versus the detuning Δ . The transition energy is scaled by the energy of the cavity ω_c and the detuning is scaled by the coupling rate g . The dashed lines indicate the bare states' transition. Note that you can somehow see a similar avoided-crossing curve in Figure 2.11 by connecting the upper (lower) dressed states in different detunings together.

Qubit-Cavity Coupled States



No detuning

$$\Delta = 0; \quad \hbar\omega_c = \hbar\omega_c; \quad \omega_c = \omega_c$$



Extra

Quantized Electric Field in Cavity

$$\hat{a} = \frac{1}{\sqrt{2\omega_c}}(\omega_c \hat{q} + i\hat{p}), \quad (2.1.6a)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\omega_c}}(\omega_c \hat{q} - i\hat{p}), \quad (2.1.6b)$$

which are annihilation and creation operators for a photon in the corresponding mode of the cavity and obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The electric and magnetic fields, which are now operators, can be represented by $\hat{a}$ and $\hat{a}^\dagger$ as,

$$\hat{E}_x(z, t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger) \sin(kz), \quad (2.1.7a)$$

$$\hat{B}_y(z, t) = i\mathcal{B}_0(\hat{a} - \hat{a}^\dagger) \cos(kz). \quad (2.1.7b)$$

$$\hat{E} \propto (\hat{a} + \hat{a}^\dagger) \propto \hat{q} \quad (2.1.10a)$$

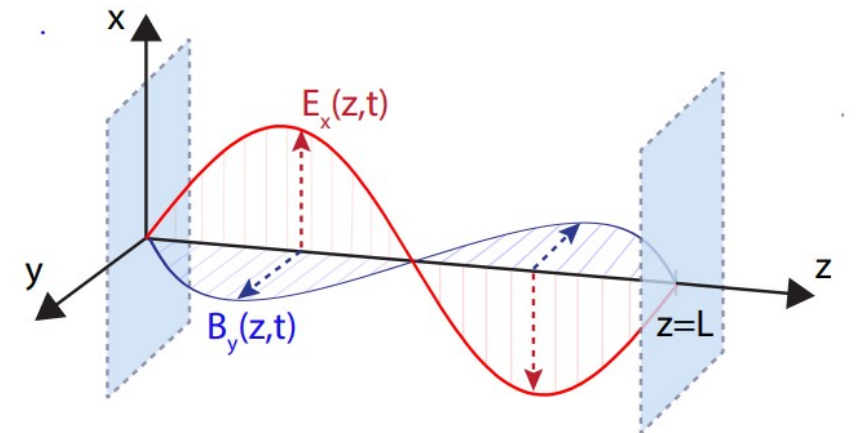
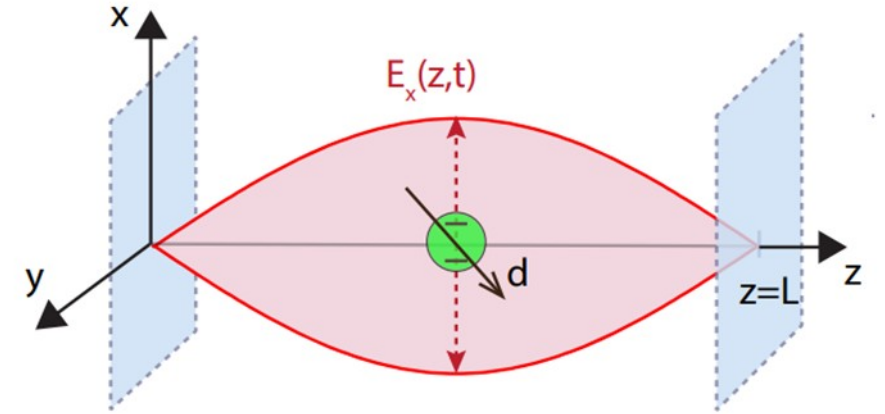
$$\hat{B} \propto (\hat{a} - \hat{a}^\dagger) \propto \hat{p} \quad (2.1.10b)$$

The Hamiltonian Eq. (2.1.5) also takes a compact representation in terms of $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{H} = \omega_c(\hat{a}^\dagger \hat{a} + \frac{1}{2}) = \omega_c(\hat{n} + \frac{1}{2}), \quad (2.1.8)$$

where the operator $\hat{n} = \hat{a}^\dagger \hat{a}$ is the *number* operator. Knowing the Hamiltonian for the electromagnetic field of a single cavity mode, we can describe the state of the cavity by solving the corresponding eigenvalue problem. Considering Hamiltonian (2.1.8) we have,

$$\hat{H}|n\rangle = E_n|n\rangle, \quad n = 0, 1, 2, \dots \quad (2.1.9)$$



Coherent state– One of the most common types of light is *coherent* light which is also known as classical light. In fact, the output of a laser or a signal generator is coherent light. Experimentally, one can simply send the output of a signal generator at the right frequency to a cavity to produce a coherent state in the cavity. The coherent state can be represented in the photon-number basis as,

$$|\psi\rangle = |\alpha\rangle = \sum_n c_n |n\rangle, \quad c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}, \quad (2.1.16)$$

where c_n indicates the contribution of each photon-number state in the coherent state. The parameter α is a constant¹ whose magnitude is related to the average number of photons, $\langle \hat{n} \rangle = |\alpha|^2$, of the coherent state $|\alpha\rangle$. In Figure 2.3 we plot c_n versus n for two different values of α .

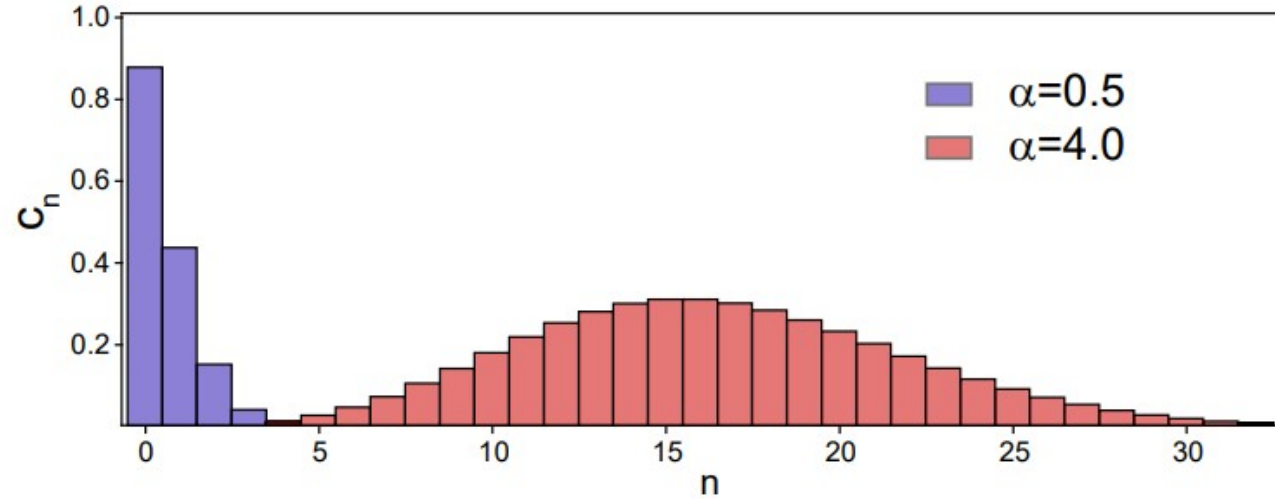


Figure 2.3: **Photon number distributions for coherent states:** The blue (red) distribution shows the photon number distribution for a coherent state which has an average photon number $\bar{n} = 1/4$ ($\bar{n} = 16$). The photon number distribution for the higher average number of photons is more like a Gaussian distribution.

Exercise 4: Show that a coherent state remains a coherent state under the time evolution but α acquires a phase: $|\alpha\rangle(t) = |\alpha_t\rangle$, where $\alpha_t = e^{-i\omega_c t}\alpha$.

$$\begin{aligned}\langle\alpha_t|E|\alpha_t\rangle &= 2\mathcal{R}[\alpha_t]\mathcal{E}_0 \sin(kz) \cos(\omega_c t) \\ &= 2|\alpha|\mathcal{E}_0 \sin(kz) \cos(\omega_c t + \phi),\end{aligned}\tag{2.1.18a}$$

$$\begin{aligned}\langle\alpha_t|B|\alpha_t\rangle &= 2\mathcal{I}[\alpha_t]\mathcal{B}_0 \cos(kz) \sin(\omega_c t) \\ &= 2|\alpha|\mathcal{B}_0 \cos(kz) \sin(\omega_c t - \phi),\end{aligned}\tag{2.1.18b}$$

It worth mentioning here that the coherent state has a very important role in quantum measurement. In particular, a precise measurement of the phase of a coherent signal is an essential component for most quantum measurement experiments. Usually, we are not interested in the natural oscillation frequency of a coherent signal. Therefore we go to a frame that exactly rotates with that frequency. In that *rotating frame*, the coherent state doesn't rotate anymore in phase space. The coherent state Wigner distribution is either along q or p or somewhere between and remains a steady-state. So in the rotating frame we freeze the time evolution for the oscillator. For simplicity let us assume the oscillator state is along the q axis, which means all the energy is potential (like a stretched spring or a pendulum at its turning point). In the rotating frame, the coherent state is stationary and the phase is fixed unless, for any reason, the coherent state experiences an external phase shift (or a kick) on top of its normal phase evolution due to a perturbative interaction. In such case, the coherent state rotates to a new place in phase space We can easily detect that displacement in the rotating frame¹(see Figure 2.5). We will see in the next chapter that this type of phase detection is the essence of qubit readout measurement.

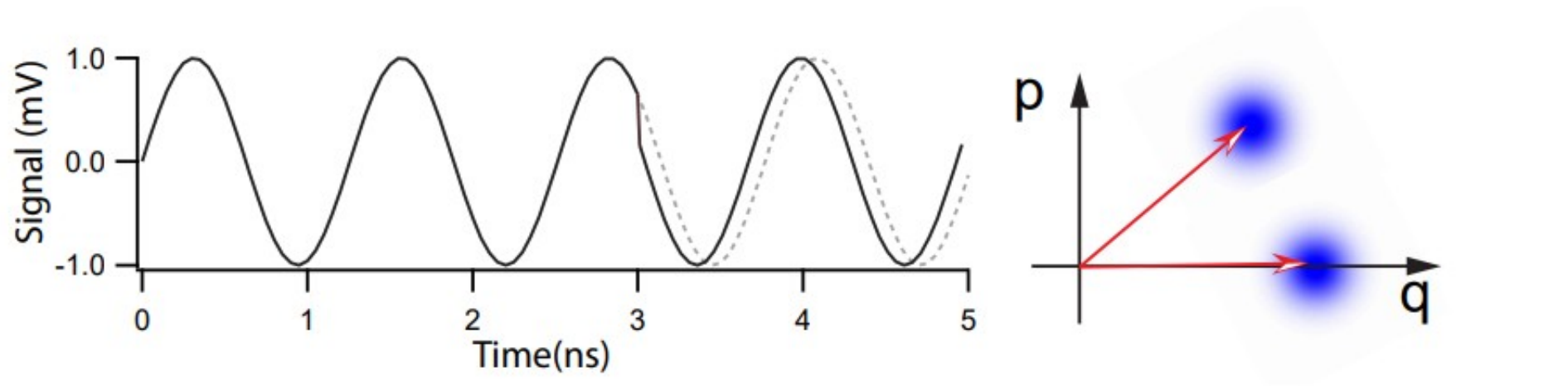


Figure 2.5: **Phase shifts for coherent state in the rotating frame:** The phase shift of a coherent signal is easily detectable in the rotating frame.

Remember $\mathbf{p}$ and $\mathbf{q}$ is similar to quadrature operators $\mathbf{X}_1$ and $\mathbf{X}_2$

$$\hat{H}_{\text{dis}} = \omega_c \hat{a}^\dagger \hat{a} - \frac{1}{2} \omega_q \sigma_z - \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \sigma_z. \quad (2.3.19)$$

$$\hat{H}_{\text{dis}} = (\omega_c - \chi \sigma_z) \hat{a}^\dagger \hat{a} - \frac{1}{2} \omega_q \sigma_z, \quad (2.3.20)$$

Alternatively, one can rearrange the terms in ((2.3.19)) as,

$$\hat{H}_{\text{dis}} = \omega_c \hat{a}^\dagger \hat{a} - \frac{1}{2} (\omega_q + 2\chi \bar{n}) \sigma_z, \quad (2.3.21)$$

and interpret the interaction as a shift in qubit frequency due to photon occupation ($\bar{n} = \hat{a}^\dagger \hat{a}$) in the cavity¹.

Generally, in the experiment we use an on-resonance drive to prepare the qubit states. Figure 2.15 summarizes our discussion of a driven qubit in the lab and rotating frame by showing Rabi oscillations of a qubit initialized in ground state for both on-resonant and detuned drives.

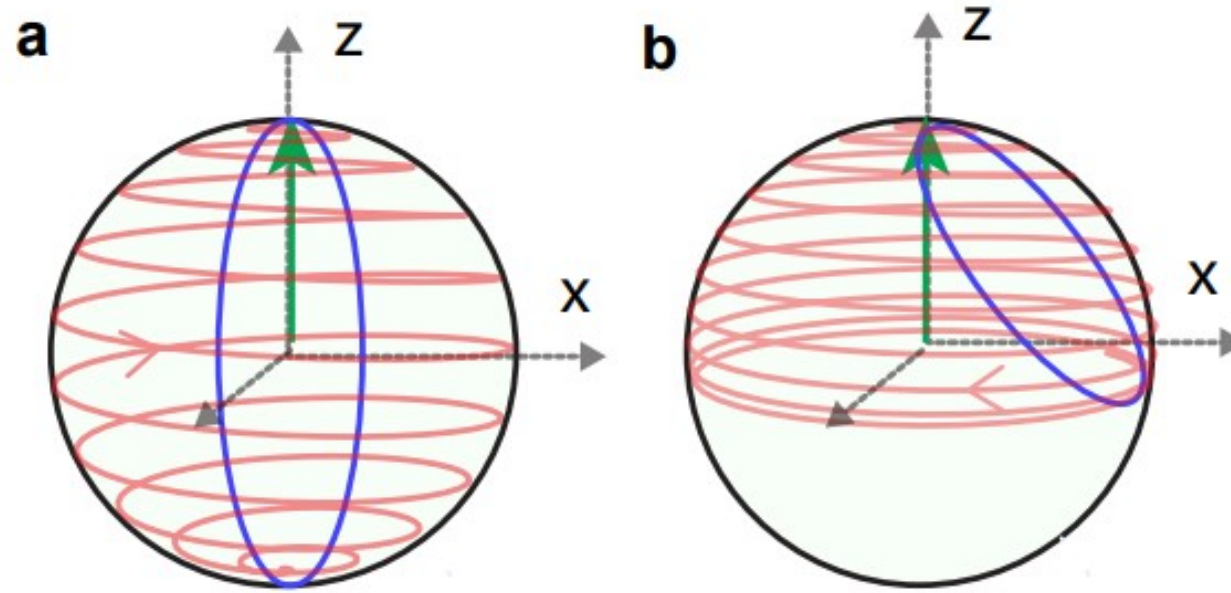


Figure 2.15: **Driven qubit state evolution in the Bloch sphere:** The red (blue) line shows the evolution of a driven qubit in the lab frame (rotating frame of the drive) **a**, for an on-resonant drive and **b**, for a detuned drive.

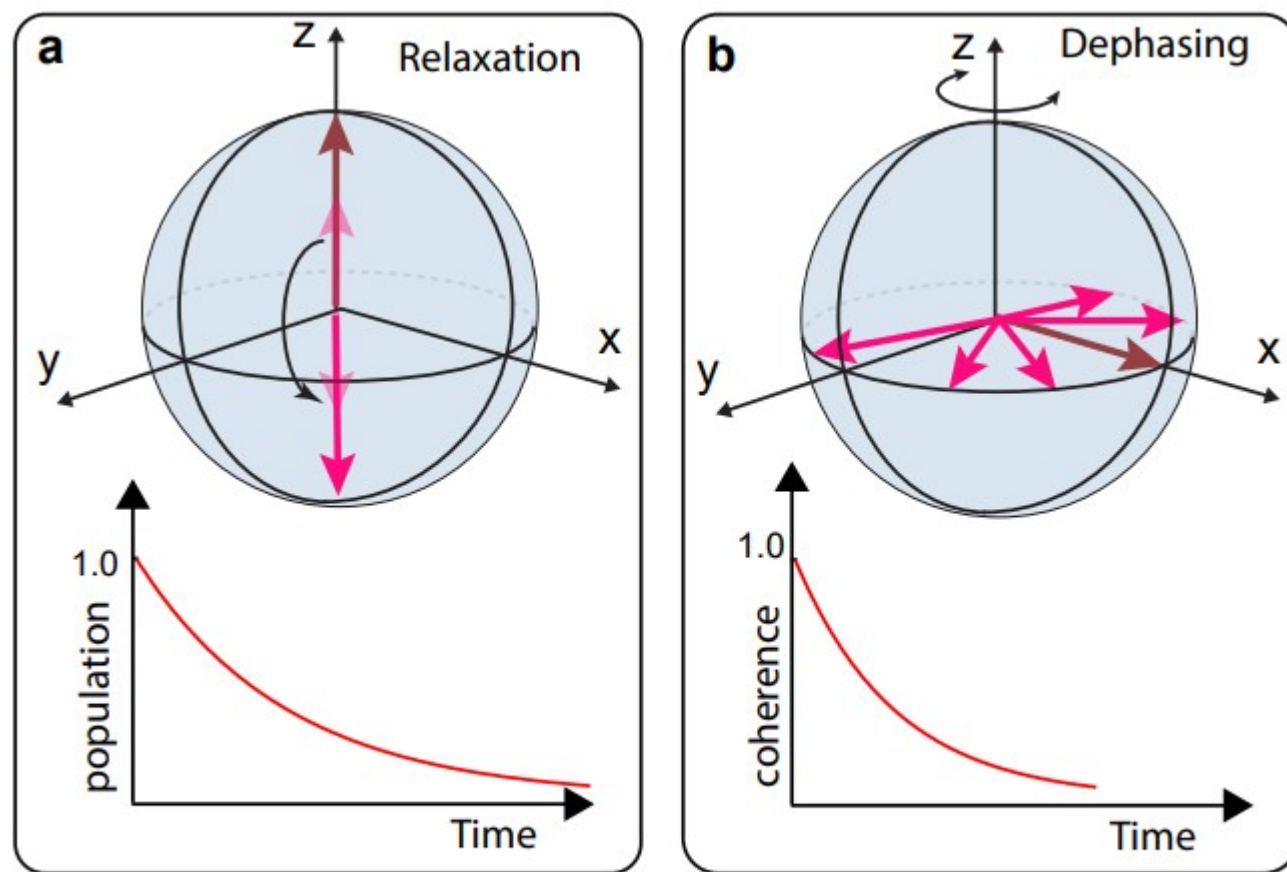


Figure 2.16: Relaxation and dephasing of a qubit.

Chapter 3

3.1 Cavity

In the previous chapter, we discussed a 1D cavity by considering two perfectly conducting walls separated by a distance L . Aluminum is a good choice for these walls since it becomes superconducting below 1.2 K and can be used to realize a high quality factor cavity. Copper can also be used when we need to thread external magnetic flux through the cavity to tune the qubit resonance.

Although we induced the cavity in 1D, it is trivial to extend the result to 3D. One can show that for a 3D cavity described by L_x, L_y, L_z dimensions and depicted in Figure 3.1, the Equation 2.1.2 simply generalized to,

$$E(\vec{r}, t) = \mathcal{E} q(t) \sin(\vec{k} \cdot \vec{r}) \quad (3.1.1a)$$

$$B(\vec{r}, t) = \mathcal{E} \frac{\mu_0 \varepsilon_0}{k} \dot{q}(t) \cos(\vec{k} \cdot \vec{r}), \quad (3.1.1b)$$

where $\vec{k} = (n_x \pi / L_x, n_y \pi / L_y, n_z \pi / L_z)$ and $\vec{r} = (x, y, z)$ and the corresponding resonance frequency of modes are,

$$f = \omega_c / 2\pi = \frac{c}{2} \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}, \quad (3.1.2)$$

where c is the speed of light inside the cavity. Each mode is described by a set of integers n_x, n_y, n_z , for example, TE_{101} corresponds to a mode with an the electric field profile that

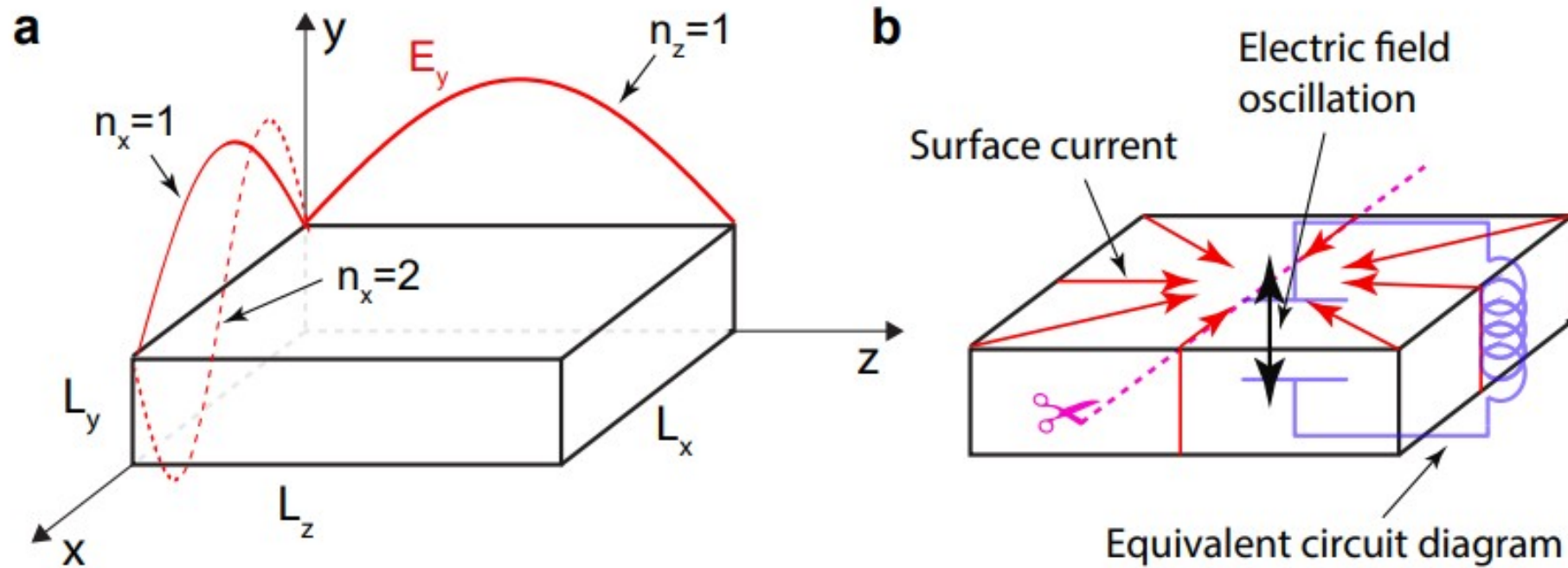


Figure 3.1: **TE_{101} mode in rectangular 3D cavity:** **a**, The typical cavity geometry is shown. Red lines shows the electric field profile along z, x spatial directions for the TE_{101} mode. The electric field oscillations are maximum at the center of the cavity. **b**, The surface current oscillations (red arrows) have been depicted for the TE_{101} mode. The induced charges are maximum at the center (opposite charges at the top and bottom of the cavity). The equivalent circuit diagram is shown for a section of the cavity. The cut-line (magenta dashed line) indicates one of the planes where the surface current is tangential.

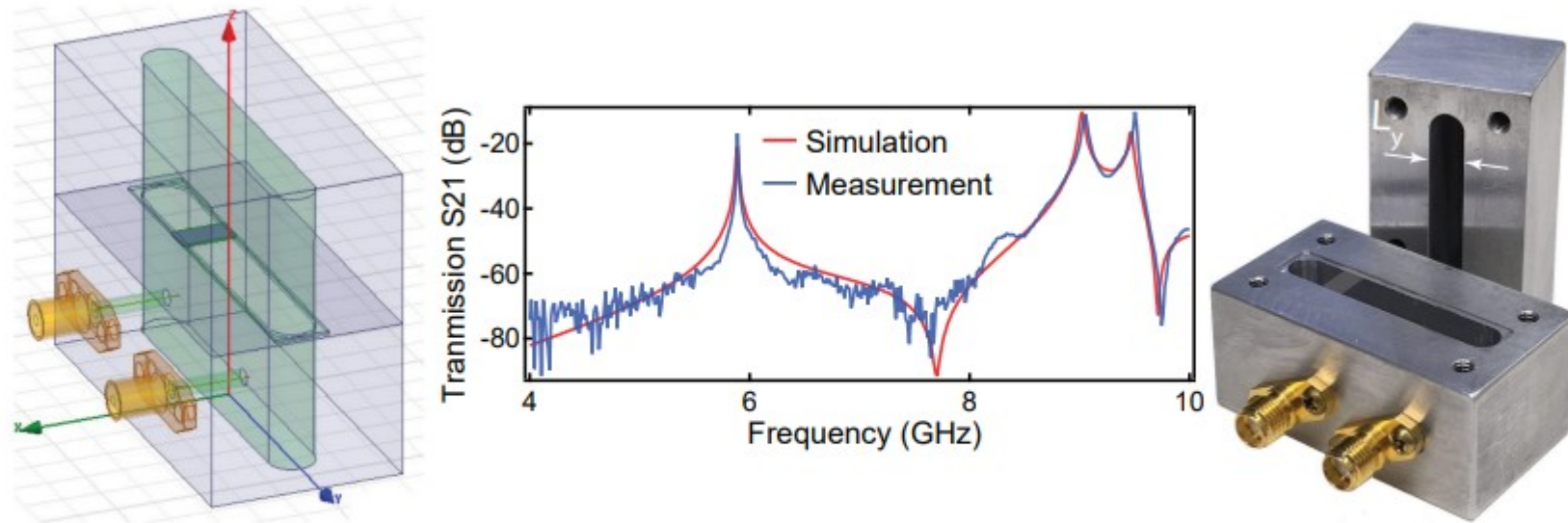


Figure 3.2: **HFSS simulation for cavity transmission:** The cavity transmission is simulated by HFSS (red curve) which is in agreement with actual measurement (blue curve).

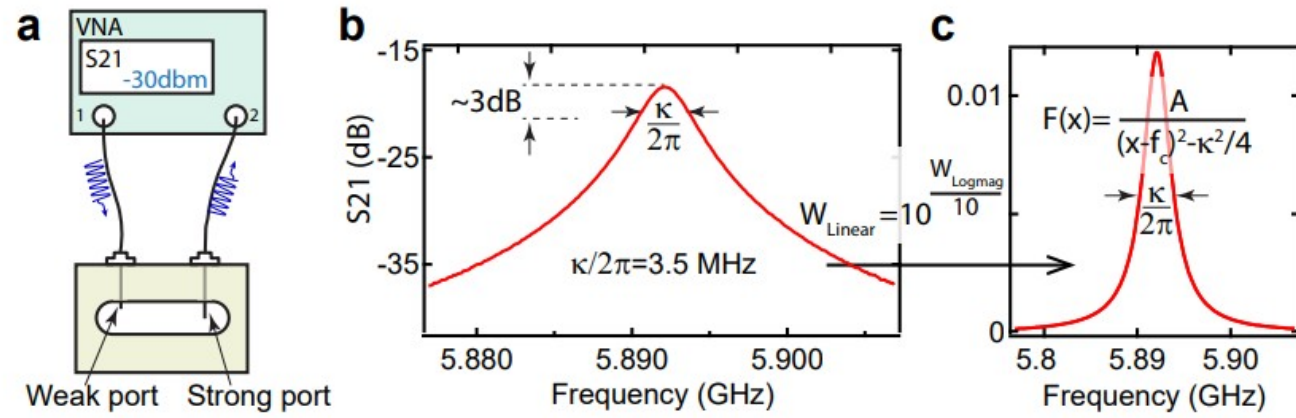


Figure 3.3: **The cavity linewidth characterization:** The cavity linewidth κ can be quantified by measuring the scattering parameters of the cavity. **a,b** In the transmission measurement S_{21} , the cavity linewidth can be estimated by the bandwidth that the transmission signal drops by 3 dB. **c**, More carefully, one can scale the transmission in the linear form and fit to the Lorentzian function. The FWHM of the Lorentzian function would be the cavity linewidth.

reflected) signal¹.

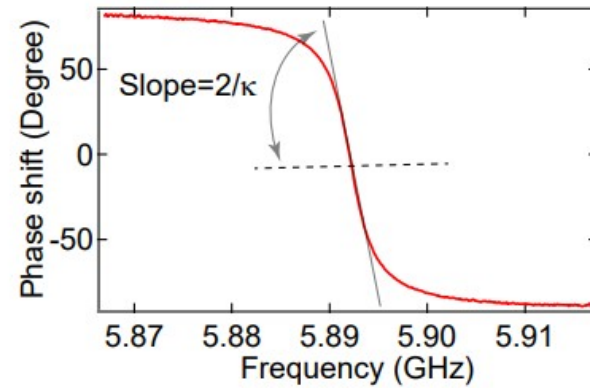


Figure 3.4: **The cavity phase shift across the resonance:** The transmitted signal through the cavity experiences a π phase shift across the resonance of the cavity. Near resonance, the phase shift is approximately linear with a slope of $2/\kappa$.

Cavity internal and external quality factors- The external quality factor Q_{ext} can be adjusted by the length of the input and output port pin antennas. Normally, we have two pins corresponding to the weak port and strong ports. The weak port is used as an input (qubit manipulation and readout) and usually has ~ 100 times weaker coupling to the cavity compared to the strong port. The lengths of the port antennas determine the coupling to a $50\ \Omega$ transmission line, determining the external quality factor Q_{ext} . The internal quality factor (often called unloaded quality factor) has to do with the losses due to the cavity itself, e.g. absorption of photons by the cavity. The total cavity quality factor is then,

$$\frac{1}{Q_{\text{tot}}} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}}. \quad (3.1.4)$$

Often, the deliberate coupling to the outside is dominant $Q_{\text{int}} \gg Q_{\text{ext}}$. Therefore the total quality factor is almost equal to the external quality factor. Note, that $Q_{\text{tot}} = \omega_c / \kappa$.

For a careful characterization of the internal quality factor and the input and output coupling strengths, one can perform reflection measurements on each port (while the other port is terminated by $50\ \Omega$). By analyzing the amplitude and phase of the reflected signals, one can obtain both the internal and external quality factors for the cavity and characterize the coupling strength for each port (see Refs. [73, 74]).

