

Quantum Hardware

PHYS 514

**Week -03 Superconductivity and Josephson
Junction**

Saleem Rao

Term 212

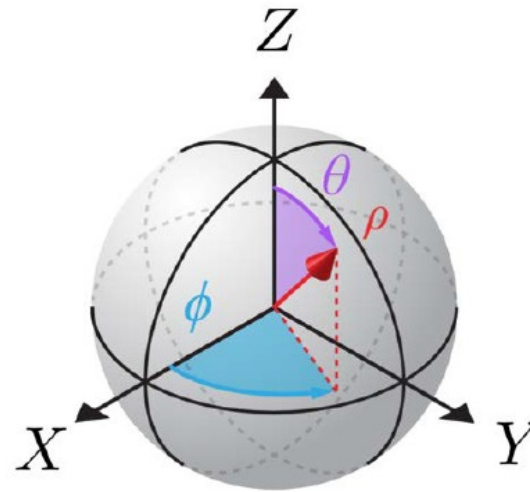
Qubit Ideally and in Reality

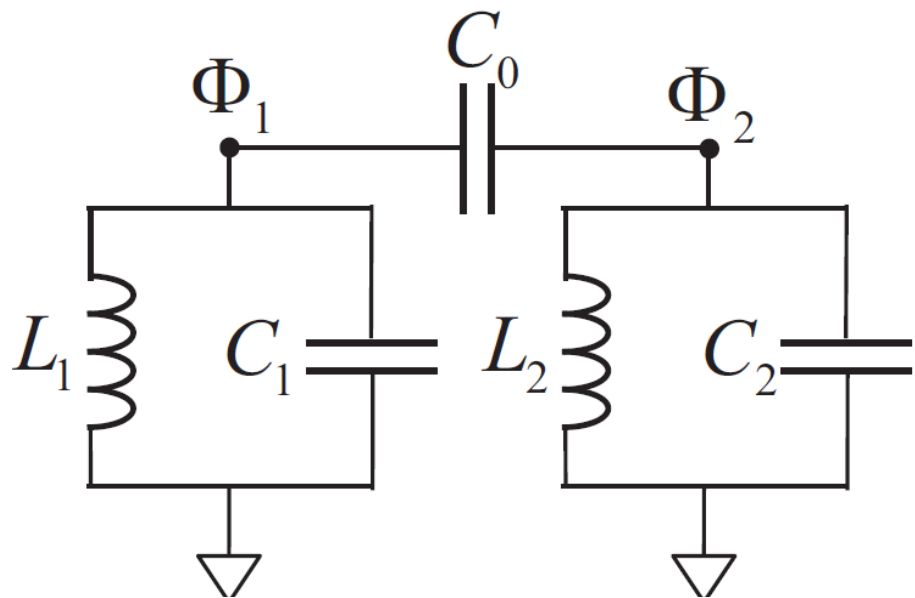
Energy levels

 $|1\rangle$

 $|0\rangle$

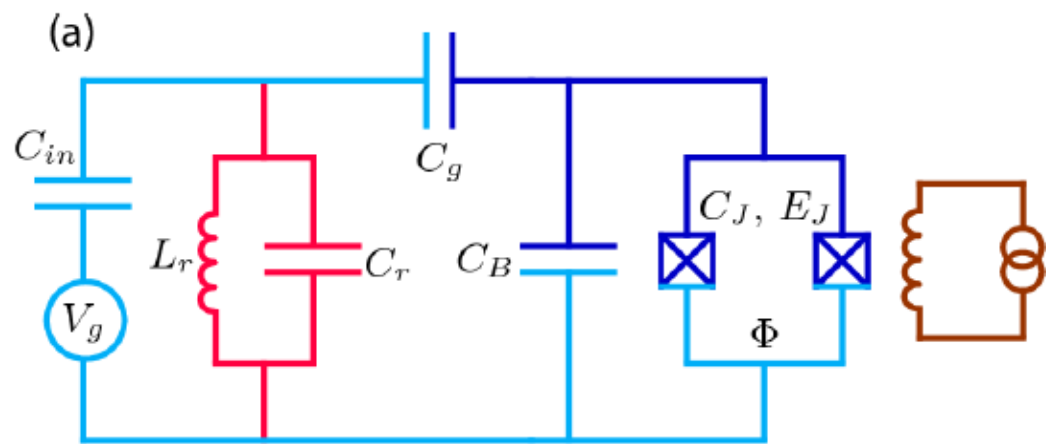
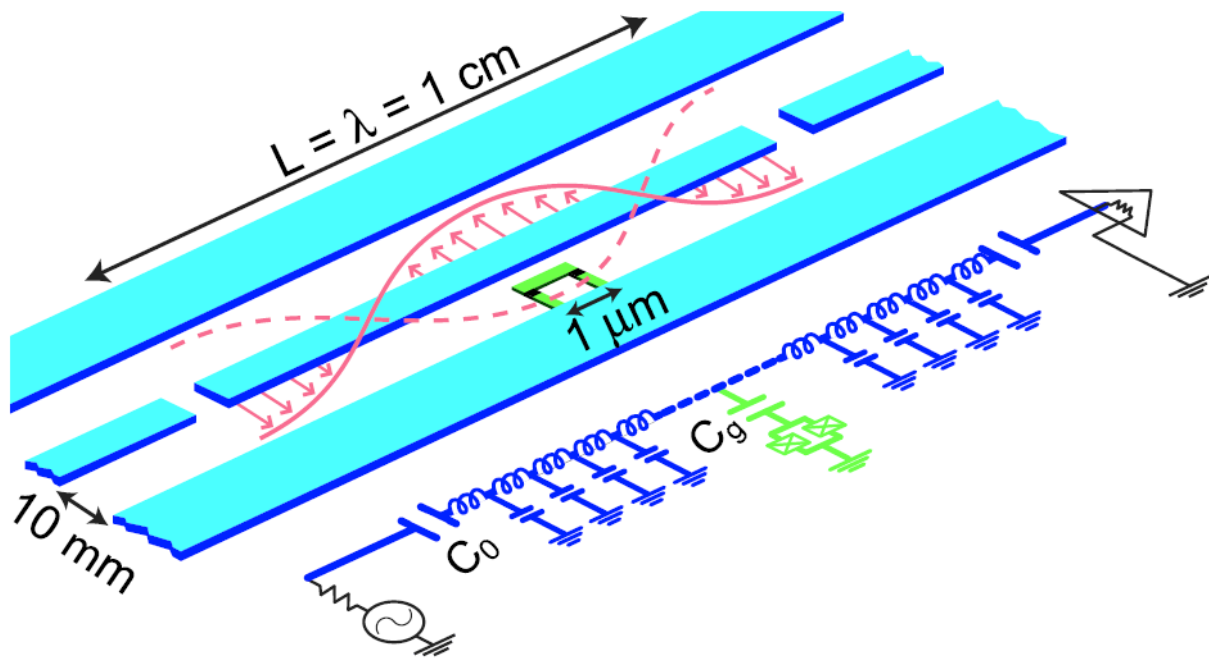
Hilbert space



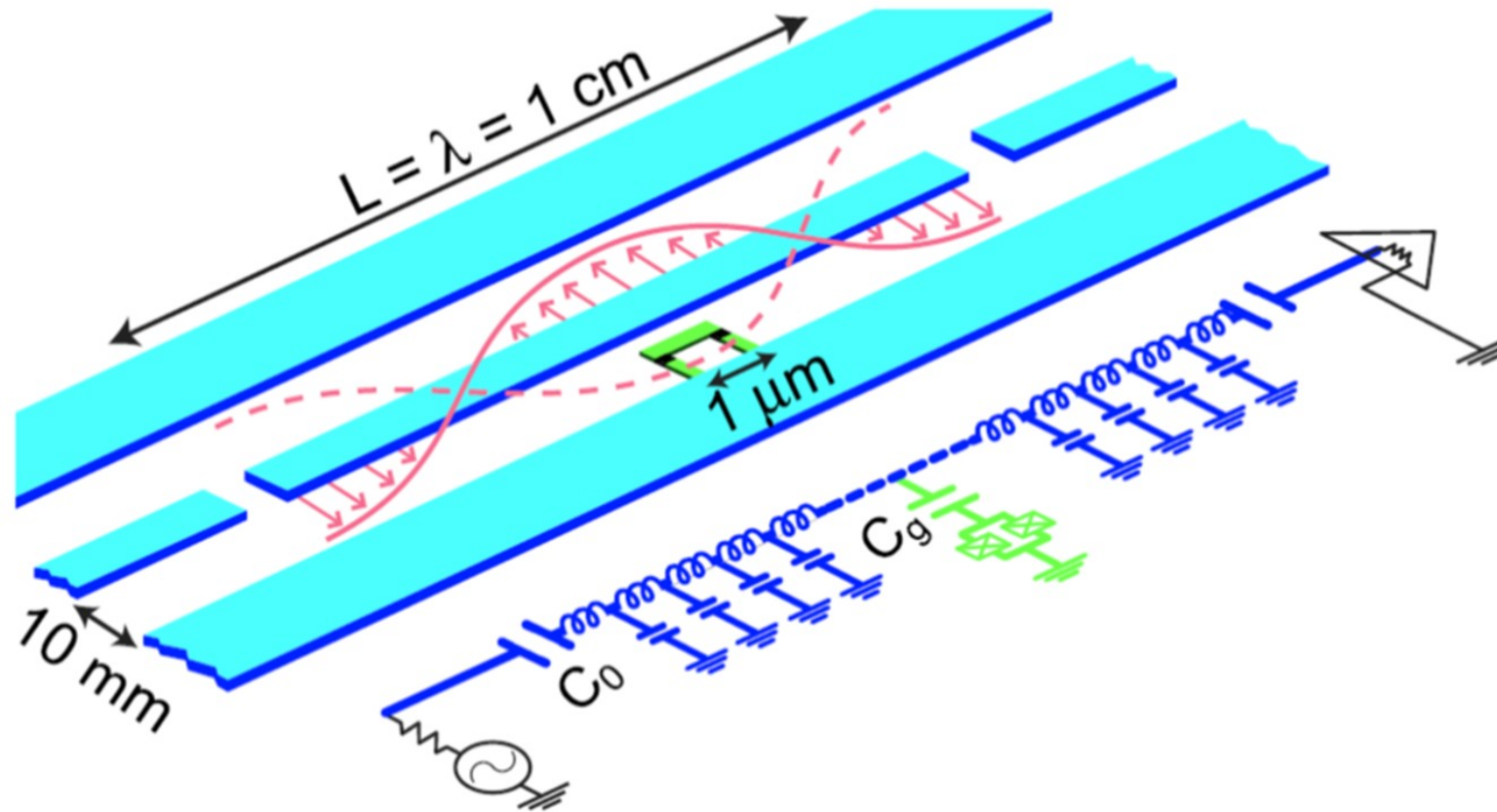


Choosing the fluxes Φ_1 and Φ_2 as the coordinates of the two oscillators, the Lagrangian can be written

$$\mathcal{L} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_0[\dot{\Phi}_1 - \dot{\Phi}_2]^2 - \frac{1}{2L_1}\Phi_1^2 - \frac{1}{2L_2}\Phi_2^2 \quad (2.110)$$



Hamiltonian of the transmission line (using Lumped elements)



L – inductance per unit length
 C – Capacitance per unit length

Coherent State of Harmonic Oscillator

$$\hat{H} = \frac{\hbar\Omega}{2} [\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger] = \hbar\Omega [\hat{a}^\dagger \hat{a} + 1/2] \quad \text{Hamiltonian of a harmonic oscillator}$$

Raising lowering operators

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

Where n is Fock state of the harmonic oscillator

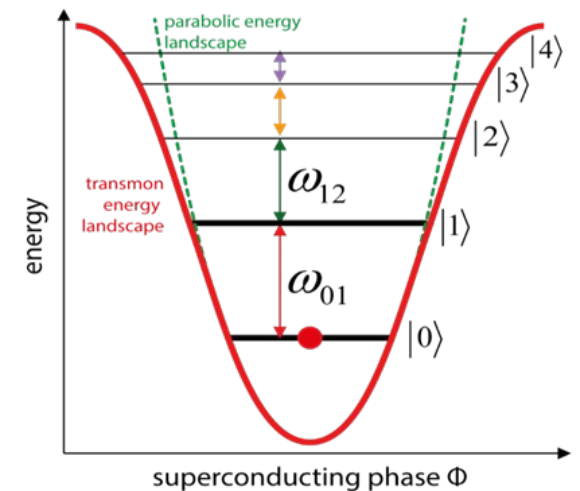
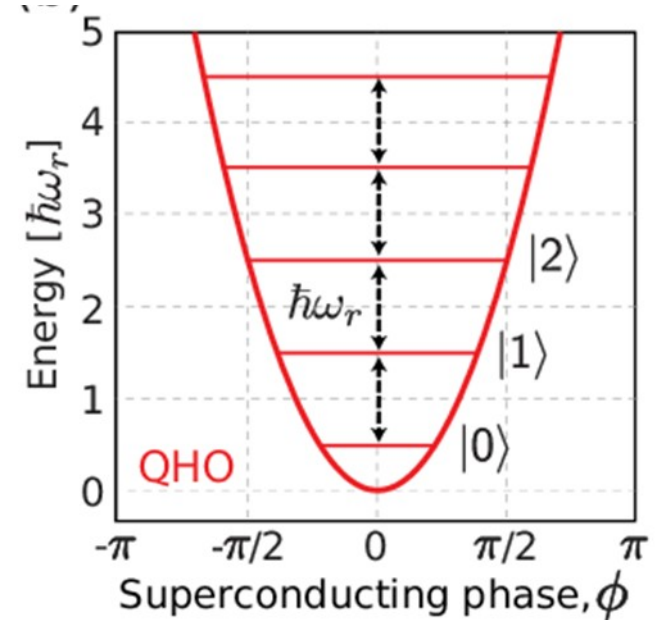
$$\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$$

$$\hat{a}^\dagger \hat{a} = \hat{n} \text{ called number operator}$$

$$\hat{n} |n\rangle = n |n\rangle$$

Coherent state $|\alpha\rangle$, is a special state of a resonator, which is very useful in **quantum optics and circuit QED**, because it behaves almost classically. It can be defined to be the eigenstate of an annihilation operator

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$$

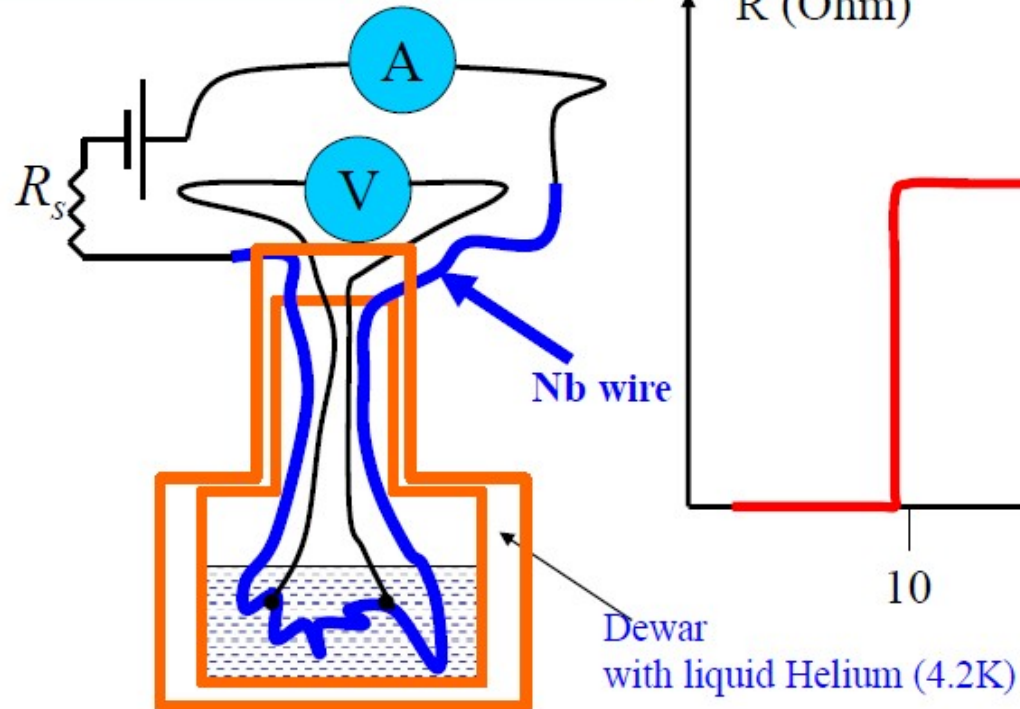


Superconductivity

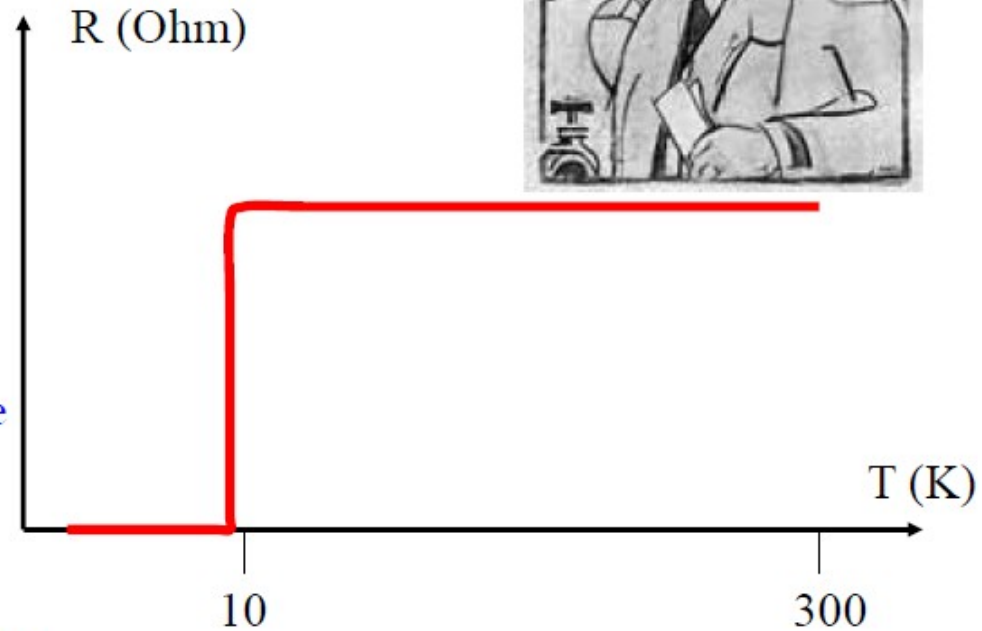
Electrical resistance of some metals drops to zero below a certain temperature which is called "critical temperature" (H. K. O. 1911)

How to observe superconductivity

- Take Nb wire
- Connect to a voltmeter and a current source
- Put into helium Dewar
- Measure electrical resistance



Heike Kamerling Onnes



Superconductivity

Superconductivity is an **extremely varied and complex phenomenon**.

The following is a brief summary of some of a superconductor's properties:

Zero resistance: Far below a critical temperature, critical magnetic field and a certain cut-off frequency, superconductors act like a perfect conductor, i.e., they conduct electricity with zero resistance.

Two-fluid behavior: In its superconducting state, a superconductor's behavior can generally be modelled as a mixture of two independent electron fluids, a normal (i.e., resistive) electron sea and an ideal superconducting fluid. The fraction of electrons participating in the normal fluid state increases with temperature until the critical temperature (T_c), where all electrons behave normally. Thus, the critical temperature is the point where we start to see superconducting effects, but they become more pronounced as the material is cooled.

Optical properties: The appearance of a superconductor doesn't change as it is cooled below its critical temperature. This means that its resistivity doesn't change at optical frequencies. This "cut-off" frequency for most superconductors is around 10^{11} Hz. It is usually called its plasma frequency.

Different metals: Not all metals appear to become superconducting at low temperatures (at least at temperatures reached to date). Metals that are superconductors are generally poor conductors in their normal state. Most good conductors (e.g., copper, gold, silver, etc.) are not superconductors.

Crystal lattice structure: The crystal structure of superconductors does not change as they enter the superconducting state.

Superconductivity

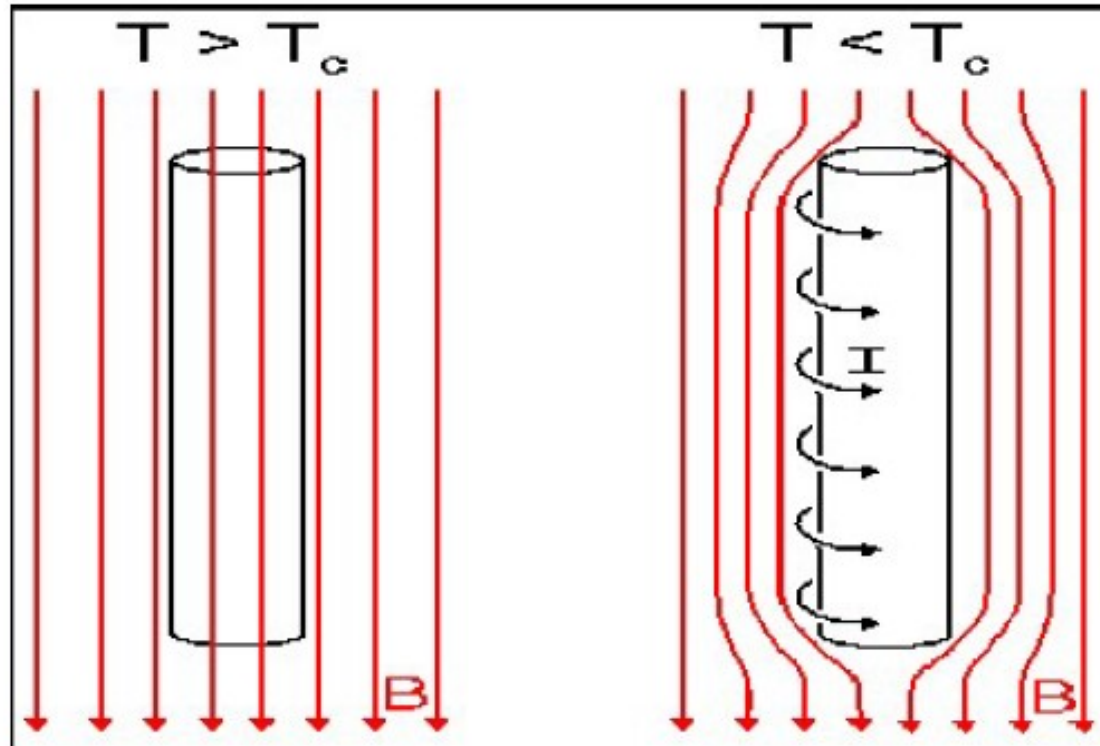
Different isotopes: Although the crystal structure of superconductors doesn't change, the critical temperature of an isotopically pure metal varies with the isotopic mass, so the mass of the lattice particles does seem to play a role.

Energy gap: At low temperatures (well below the critical temperature), the contribution to specific heat from the conduction electrons in a superconductor decays exponentially with increasing temperature, i.e., $C \sim \exp(-b/kT)$. This is consistent with the heating, dissipative process being separated by an energy gap from a superconducting ground state.

Meissner effect: A superconducting material does not just resist changes in magnetic field the way a normal (or even perfect) conductor does. It actually expels all magnetic field completely as it becomes superconducting. All electric and magnetic fields decay exponentially inside a very small surface depth known as the penetration depth. A related effect is that currents can only flow on the surface of a superconductor (again within the penetration depth of the material's surface).

Long-range coherence: The phase of the conduction electrons in a superconductor is coherent over a large distance, much larger than their typical interaction length and potentially macroscopic, even on the size scale of an electronic circuit.

Meissner effect



List of materials with their critical temperature and critical field

Formula	T_c (K)	H_c (T)	Type	BCS
Elements				
Al	1.20	0.01	I	yes
Cd	0.52	0.0028	I	yes
Diamond:B	11.4	4	II	yes
Ga	1.083	0.0058	I	yes
Hf	0.165		I	yes
α -Hg	4.15	0.04	I	yes
β -Hg	3.95	0.04	I	yes
In	3.4	0.03	I	yes
Ir	0.14	0.0016 ^[7]	I	yes
α -La	4.9		I	yes
β -La	6.3		I	yes
Mo	0.92	0.0096	I	yes
Nb	9.26	0.82	II	yes
Os	0.65	0.007	I	yes

Superconductivity and Circuit QED

Superconductivity was first discovered experimentally in 1911, but it was almost 50 years later, in 1957, before a detailed quantum theory was proposed which could explain the phenomenon properly at a microscopic level. Discovered by Bardeen, Cooper and Schrieffer, this is generally referred to as the **BCS theory of superconductivity**

More recently, however, in the 1980s, it was discovered that such a full microscopic theory is not necessary in order to explain the quantum behavior of an electronic circuit. Instead, it turns out that it is only necessary to describe the behavior of the collective macroscopic degrees of freedom which are related to the standard variables of **classical electronic circuit theory**. As a result, for the purpose of circuit QED, many of the subtleties of superconductivity are not really relevant.

Superconductivity and Circuit QED

The main simplifying factor for circuit QED is that **experiments are run at very low temperatures**, much less than the critical temperature, $T \ll T_c$. This means that **any “normal conductor” effects are negligible and the circuits behave like ideal superconductors**.

Zero Resistance

The most important reason why superconductors are used in circuit QED is that they provide a way to build resistance-free, and therefore energy dissipation free electronic circuits. Because of the very low operating temperatures, the ideal superconductivity allow the circuits to **behave as good quantum systems with long coherence times and the usual classical electronic variables can be treated quantum mechanically**.

Energy Gap

The existence of an energy gap between the superconducting ground state and the dissipative “normal” conduction processes, again combined with operating temperatures far smaller than this gap, makes these circuits substantially resistant to thermal noise, because thermal processes are unable to excite dissipation.

Optical Properties

Because the **superconducting behaviour only survives below a certain cut-off frequency**, **this limits circuit QED experiments to operating far below this frequency**. Such experiments are usually carried out in the range 1–30 GHz (and more commonly below 15 GHz). This brings some disadvantages. In fact, **working in this microwave regime** sets a much more **stringent requirement** on low-temperature operation than the superconducting critical temperature. For example, at around 170 mK, the peak of the black-body radiation spectrum is at around 10 GHz, and at 20 mK, the peak of the black-body spectrum is around 1.2 GHz. This is one motivation for working in cavities, since this limits the bandwidth of thermal radiation that is seen by the superconducting qubits.

Superconducting Condensate and Cooper Pairs

The electrons fill up the available states one at a time up to an energy known as the Fermi level, the point where all the electrons are used up.

In a bulk material, the available density of states is essentially **continuous** and **this is kind of like filling a glass with a fixed amount of water**, and this is called a “sea” of electrons, or more specifically the **Fermi sea**

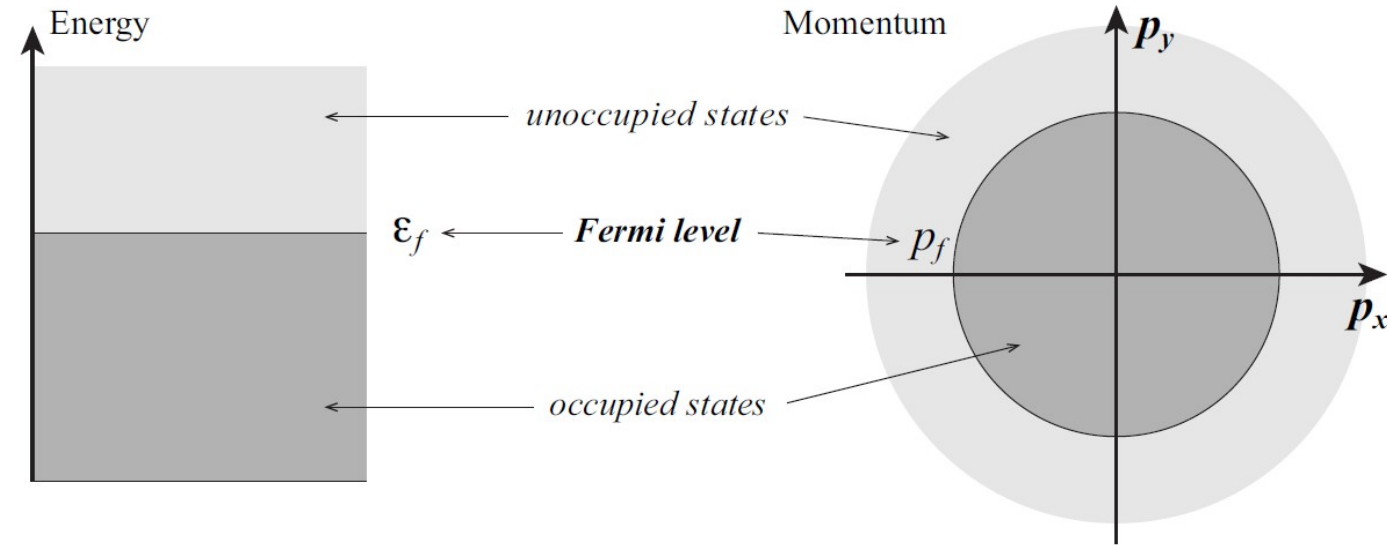


Figure 3: *Electronic band structure of a normal metal.*

In a **perfect lattice**, such **electron waves should be able to flow through the metal without losing any energy**. This is obviously not true in the real world, however. In real, imperfect lattices, electrons experience a resistance (and thus energy dissipation) because of continuous inelastic scattering events off irregularities in the lattice. These irregularities can be caused by **either lattice vibrations (phonon scattering) or impurities**.

Since the background **lattice phonons are thermally excited**, they disappear as the metal is cooled towards absolute zero temperature. Thus, a pure metal would become a perfect conductor at zero kelvin. **In an impure metal, the resistance would instead approach a constant background value caused by the lattice impurities.**

Superconducting Condensate and Cooper Pairs

In the right conditions, it turns out that **fermionic electrons can instead start behaving like bosons**. Specifically, pairs of electrons (fermions) with **opposite spins can form a bound**, bosonic “molecule”, known as a **Cooper pair**, which has spin 0 (spin $+1/2$ + spin $-1/2$).

The attractive force which binds this molecule is induced by an exchange of phonons (vibrational lattice excitations) between two phonon scattering events.

Interestingly, these scattering events, which are responsible for causing resistance in normal metal conduction, are now a key ingredient in superconducting behaviour. This is why superconductors are poor conductors when in the normal state.

Understanding the details of the **microscopic theory of superconductivity** is generally almost entirely unnecessary for understanding circuit QED. Ignoring the details of this pairing process, therefore, the key take-away point is the fact that these **paired electrons are able to behave completely differently from unpaired electrons**, which are subject to all the usual constraint imposed by the Pauli exclusion principle.

Superconductor's electrons form a giant condensate, (Cooper pair are electrons with opposite spin and equal and opposite momenta), this condensate represents a **macroscopic collective ground state for the superconducting system**.

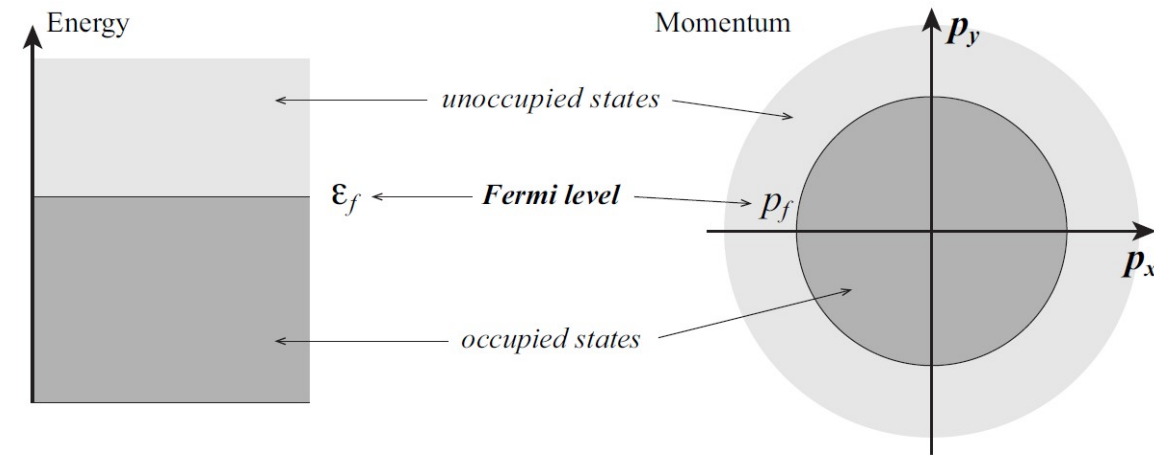


Figure 3: *Electronic band structure of a normal metal.*

Electron is particle or Wave

Electron is a particles

No need to discuss because you all have good idea about this concept

Electron is wave ?

What is this!

if electron is a wave
then what is 'waving' in this case

Schrodinger's Wave Equation

$$-\frac{\hbar}{2m} \frac{d^2}{dx^2} \psi(x) + V\psi(x) = i \hbar \frac{\partial}{\partial t} \psi(x)$$

What is ψ

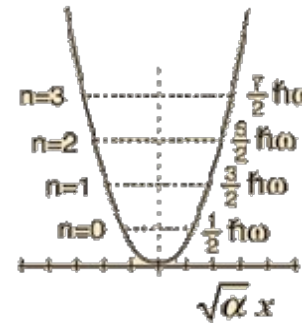
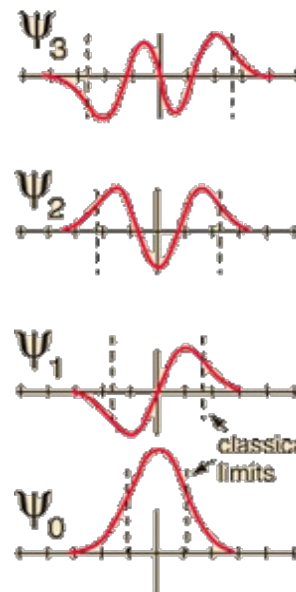
$\psi(x)$ is Wave function

~~$\psi(x)$ – electron charge density !~~

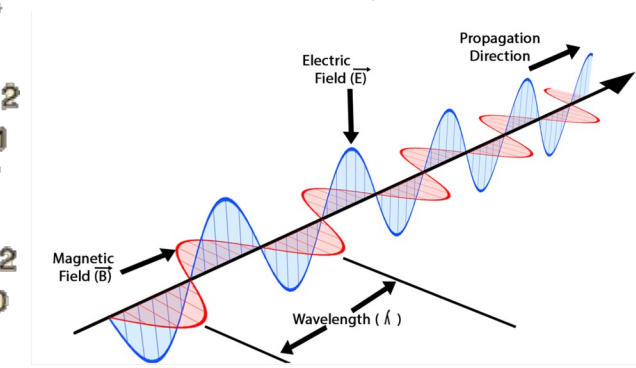
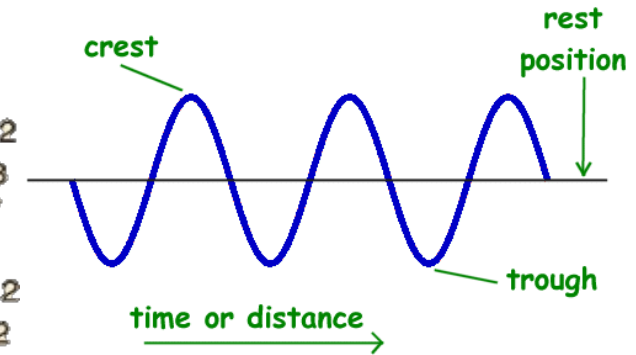
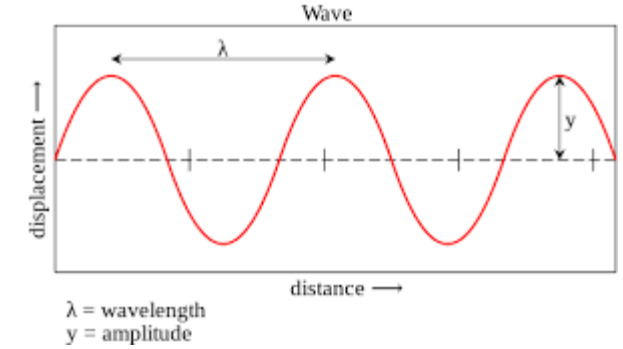
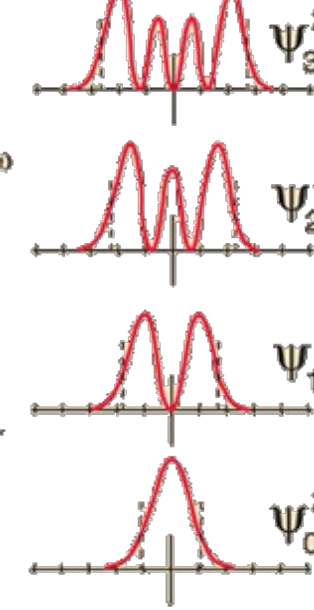
$\psi(x)$ – probability density
of finding the particles

Schrodinger's

Max Born



Harmonic oscillator
potential and
wavefunctions



Wave function $\psi(x)$ has no physical meaning, $|\psi(x)|^2$ is the probability of finding the particle at location 'x'

What is $\psi(x)$ or $|\psi\rangle$ and how can find it

$\psi(x)$ or $|\psi\rangle$ is the wave function of the quantum system

Wave function $\psi(x)$ can be determined using Schrodinger 's wave equation

$$-\frac{\hbar}{2m} \frac{d^2}{dx^2} \psi + V\psi = i \hbar \frac{\partial}{\partial t} \psi$$

If V is **time independent**, above equation can be solved by **separating the variable technique** i.e. position dependence of $\psi(x)$ (that can be 3D) can be determined independently and later time dependence can be determined using unitary time evolution operator (we already talked about for harmonic oscillators)

$$\hat{H}|\psi\rangle = E|\psi\rangle \quad \text{If we know Hamiltonian } \hat{H}, \text{ we can find the wave function } |\psi\rangle$$

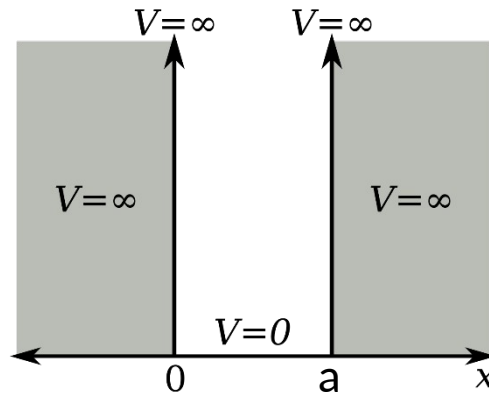
For LC oscillator (including resonators) we know the recipe to find the Hamiltonion

Can we use same for Superconductors ? Specially for Josephson Junction

Wave Function $|\psi\rangle$ for Particles in Infinite Well

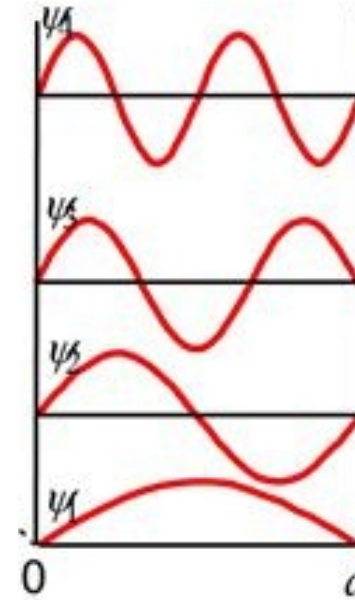
What are the possible energy states of an electron in this infinite potential well

Infinite potential well

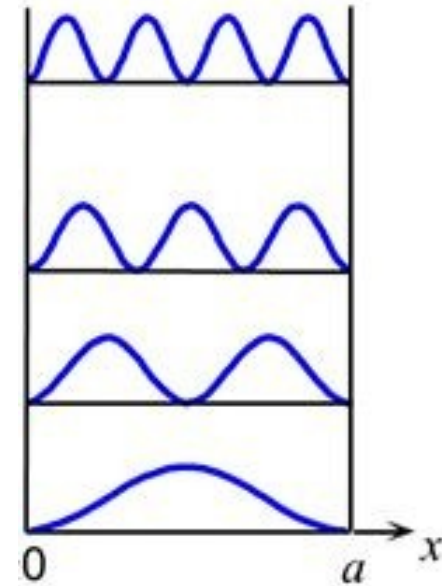


By solving following SWE for $V=0$ we can find the $\psi(x)$ for given infinite potential well

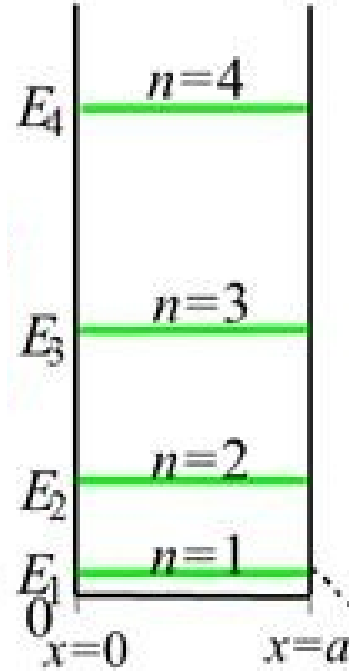
$$\frac{d^2}{dx^2} \psi(x) + \frac{2m}{\hbar^2} (E - V) \psi(x) = 0$$



Wave function is **entirely confined** in the potential well



$|\psi(x)|^2$
probability



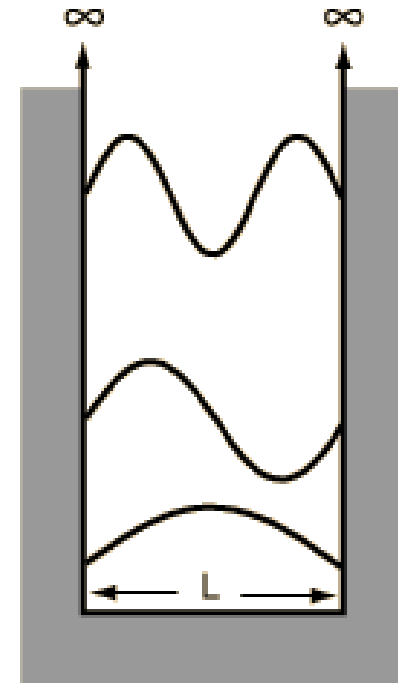
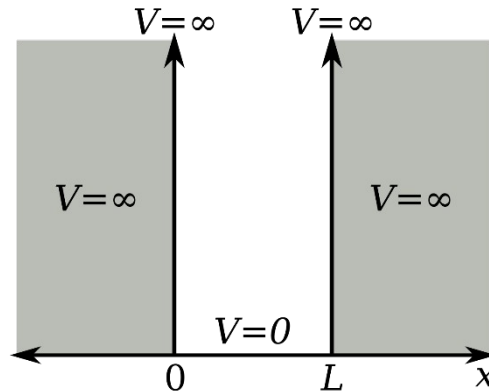
Possible
energy states

Wave Function $|\psi\rangle$ for Particles in Infinite and Finite Well

Wave function in infinite and finite potential well

What are the possible energy states of an electron in this infinite potential well

Infinite potential well

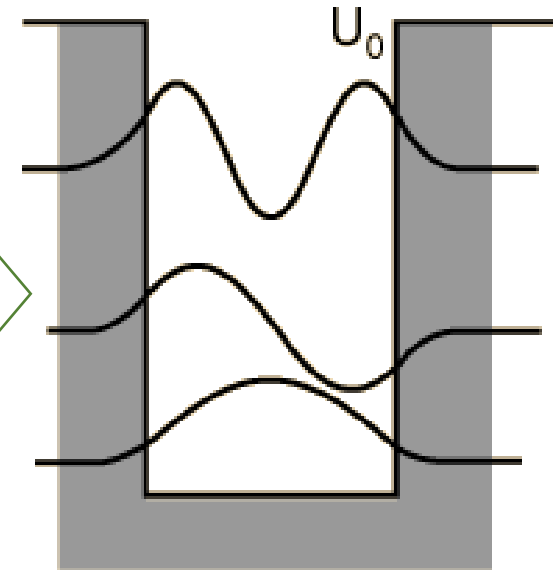


Barrier height is ∞

Wave function is **entirely confined** in the potential well

Probability of finding the electron **outside well is zero**

If barrier height is finite



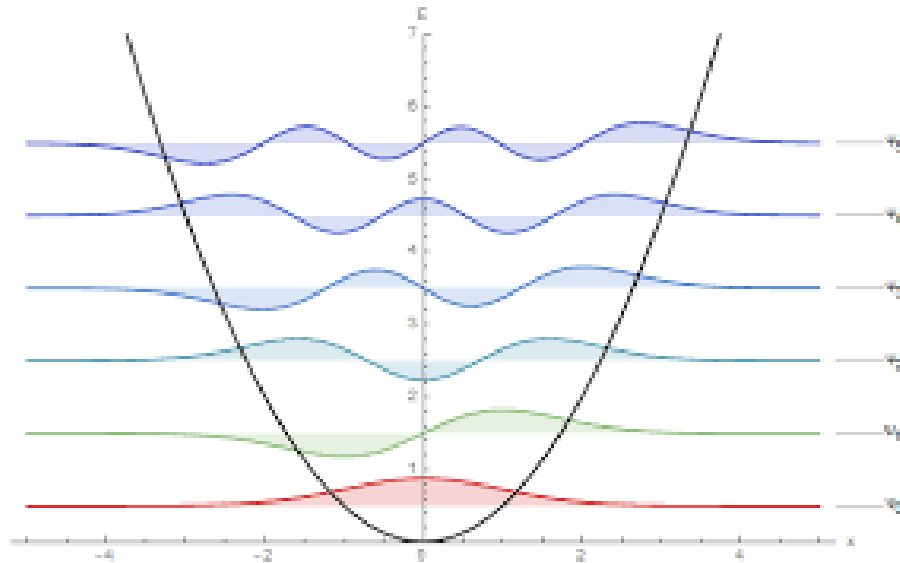
Barrier height is finite U_0

Wave function is **extends** out of the potential well

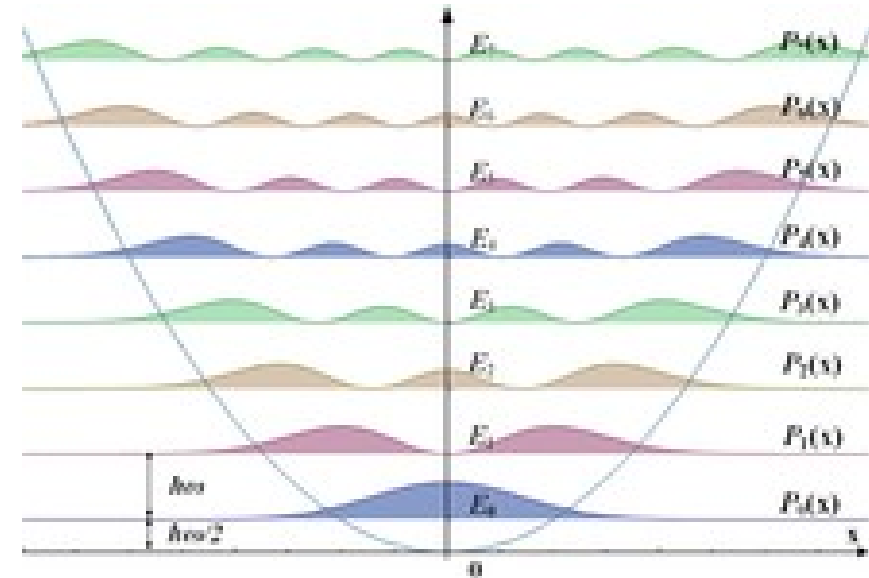
Probability of finding the electron **outside well is NOT zero**

Wave Function $|\psi\rangle$ for Harmonic Oscillator

Wave function $|\psi\rangle$



$|\psi(x)|^2$: probability



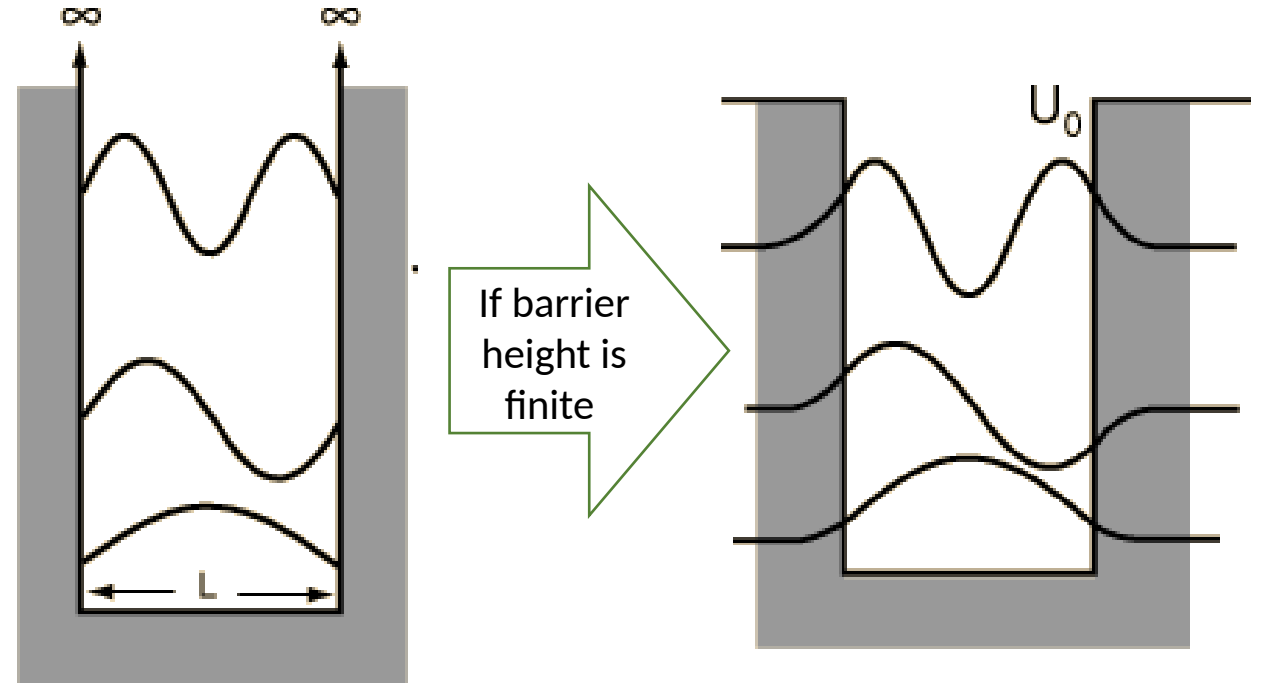
Remember we did not talk about the mathematical details of the wave function

Wave Function $|\psi\rangle$ for Fermions and Bosons

Electrons are the fermions

only two electrons (with opposite)
can occupy each energy state

Cooper pair (pair of opposite spin
electrons) **are Bosons**, **all can be in
same energy state**



All cooper pair can defined by the same wave function!

$$|\psi\rangle = \sqrt{n_s} e^{i\varphi}$$

φ is called phase factor

Superconducting Condensate and Cooper Pairs

In the superconducting condensate, each Cooper pair occupies a wave-function of the following form:

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = \sum a_j \phi(\mathbf{r}_1, \mathbf{P}_j, \uparrow) \phi(\mathbf{r}_2, -\mathbf{P}_j, \downarrow)$$

$|\Phi(\mathbf{r}_1, \mathbf{r}_2)|^2 =$ is the prob density at $\mathbf{r}_1$, and $\mathbf{r}_2$,

and $\phi(\mathbf{r}, \mathbf{P}, \uparrow)$ is single electron wave function with momentum $\mathbf{P}$ and spin in the "up" direction

This state is a superposition of states with many different momenta and this is where some of the unique properties of the superconductor come from. As described above, the resistance in normal metals results from the fact that electrons continuously scatter off lattice irregularities. **In superconductors, the electrons still undergo these same scattering events, but within each pair, the phonon that is created at one electron is immediately absorbed by the other.** Because total momentum is conserved during these interactions, this just converts between different terms on the right of the above wave-function. In some sense, because all of these different states are already present in the large superposition anyway, **these continuous scattering events don't change the overall quantum state of the pair and no energy is lost from the system.**

Essentially, the strong symmetry of the Cooper-pair wave function enforces symmetric scattering events. Creating an excitation in the superconducting ground state requires breaking this symmetry in the form of breaking a Cooper pair and creating two individual unpaired electrons, which therefore again obey fermionic statistics. The energy barrier that must be overcome to do this therefore creates an energy gap between ground states and excited states.

Superconducting Phase and Current Flow

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = \sum a_j \phi(\mathbf{r}_1, \mathbf{P}_j, \uparrow) \phi(\mathbf{r}_1, -\mathbf{P}_j, \downarrow)$$

This wave-function describes a perfectly isotropic distribution of electron momenta, where for every electron travelling with a momentum $\mathbf{P}$, there is another travelling at $-\mathbf{P}$. This describes a superconductor “at rest”, where there is no current flow.

Now, if we **apply an external field to the superconductor**? In this situation, it turns out that we can write down an alternative wave-function:

$$\Phi_P(\mathbf{r}_1, \mathbf{r}_2) = \sum a_j \phi(\mathbf{r}_1, \mathbf{P}_j + \mathbf{P}/2, \uparrow) \phi(\mathbf{r}_1, -\mathbf{P}_j + \mathbf{P}/2, \downarrow)$$

This wave-function retains all of the collective symmetry properties which underlie the superconducting behaviour, but each Cooper pair now carries a resultant momentum $\mathbf{P}$, *which is the same for all pairs and all states in the wave-function*.

Because of the plane wave-like form of the underlying single-electron wave-functions $\phi \propto \exp(i\mathbf{P} \cdot \frac{\mathbf{r}}{\hbar})$, it is therefore possible to combine the contributions from the overall momentum in a global term outside the summation:

$$\boxed{\Phi_P(\mathbf{r}_1, \mathbf{r}_2) = \sum a_j \phi(\mathbf{r}_1, \mathbf{P}_j, \uparrow) \phi(\mathbf{r}_1, \mathbf{P}_j, \downarrow) e^{i\mathbf{P} \cdot \mathbf{r}_1 / 2\hbar} e^{i\mathbf{P} \cdot \mathbf{r}_2 / 2\hbar}} \quad \Rightarrow \quad \boxed{\Phi_P(\mathbf{r}_1, \mathbf{r}_2) = \Phi(\mathbf{r}_1, \mathbf{r}_2) e^{i\mathbf{P} \cdot \mathbf{r} / \hbar}}$$

where $\mathbf{r} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2$
Position of cooper pair CoM

Thus, each pair can be treated as a single “particle” with momentum $\mathbf{P}$ at its center-of-mass position $\mathbf{r}$, with mass $2m$ and charge $2e$.

Superconductors and Cooper Pair

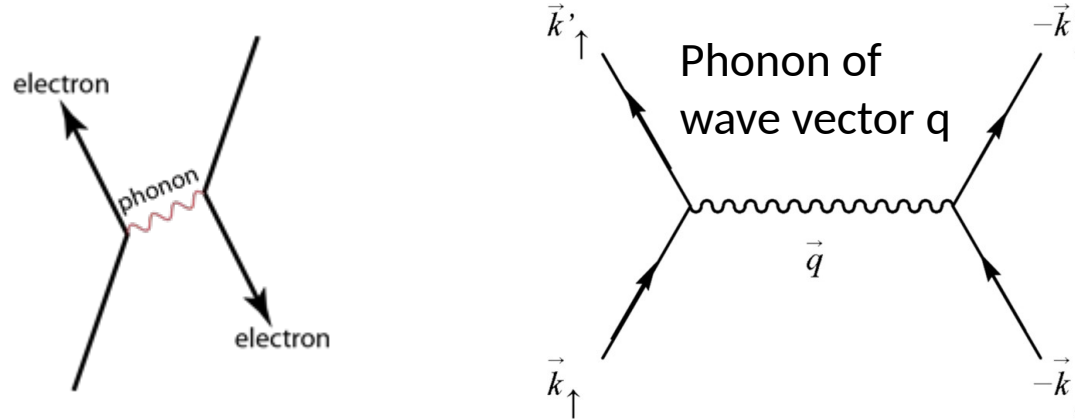
Screened columbic repulsion + Phonon mediated attraction between two electron

Below

Critical temperature

Critical magnetic field

Cooper pair forms due to phonon interaction



Attractive phonon mediated electron-electron interaction

An 'electron- electron' molecules move together

'Electron-electron' pairing is keep on changing in superconductor

Ground state: Superfluid pair ground state

“**pair of electron** – **Cooper pair**”, are **boson** and they share one quantum state, therefore one wave function can define their quantum state

$$\psi(\varphi) = \sqrt{n_s} e^{i\varphi}$$

φ is the phase factor
 n_s is density of cooper pair

What Phase Factor and Its 'Rate of Change'

$$\psi(\varphi) = \sqrt{n_s} e^{i\varphi}$$

φ is the phase factor

n_s is density of state of cooper pair

Phase evolution equation: $2eV = \hbar \frac{d\varphi}{dt}$

Where V is Voltage across the Superconductor

Remember Schrodinger equation $i\hbar \frac{d\psi}{dt} = E\psi$ ($\hat{H}\psi = E\psi$)

Thus the phase is $\varphi = \frac{Et}{\hbar}$

The energy of a pair of superconducting electron is $E = 2eV$ Where V is electric potential

Thus we obtain the phase evolution equation $2eV = \hbar \frac{d\varphi}{dt}$

Quanta in Circuits

- Charge quanta

e – charge on Electron (in case of SC: $2e$)

- Flux quanta

$$\Phi_0 = \frac{h}{2e} - \text{Flux quanta}$$

- Resistance Quanta

$$R_Q = \frac{h}{(2e)^2} - \text{Resistance quanta}$$

What Phase Factor and magnetic flux of the superconducting loop: **Flux Quanta**

Quantum state of cooper pairs

$$\psi(\varphi) = \sqrt{n_s} e^{i\varphi}$$

φ is the phase factor

n_s is density of state of cooper pair

We know superconducting state (of all cooper pair i.e. Bosons) can be defined by one wave function

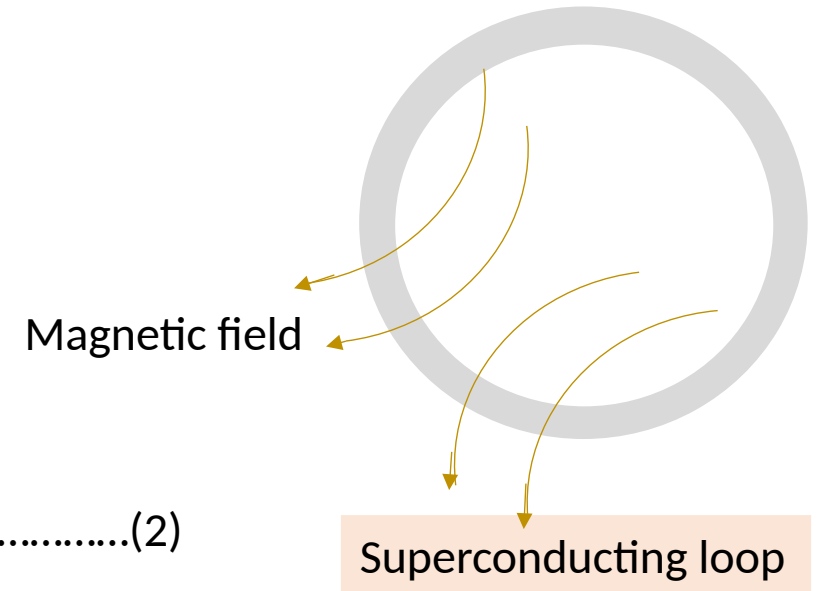
$$\psi(\varphi) = \sqrt{n_s} e^{i\varphi}$$

Further wave function must be single valued at each point, therefore

$$\varphi = 2n\pi \quad \dots\dots\dots(1)$$

From magnetic vector potential

$$\varphi = \frac{q}{\hbar} \oint \vec{A} \cdot d\vec{l} = \frac{q}{\hbar} \int \vec{\nabla} \times \vec{A} \cdot d\vec{s} = \frac{q}{\hbar} \int \vec{B} \cdot d\vec{s} = \frac{q}{\hbar} \phi \quad \phi - \text{magnetic flux} \dots\dots\dots(2)$$



From eq. (1) and (2)

$$\frac{q}{\hbar} \phi = 2n\pi$$



$$\phi = \frac{\hbar}{q} 2n\pi$$



$$\phi = n \frac{h}{q}$$

$$(h = 2\pi\hbar)$$

For superconductors $q = 2e$

$$\phi = n \frac{h}{2e}$$



$$\phi = n\Phi_0$$

$$\Phi_0 = \frac{h}{2e} \text{ called 'flux quanta'}$$

$$\Phi_0 = 2.067833848... \times 10^{-15} \text{ Wb}$$

Extremely small number !

Magnetic flux is quantized in superconducting loops

Normal Metal Tunnel Junction with equilibrium Fermi Energy ϵ_F and Applied Voltage bias V

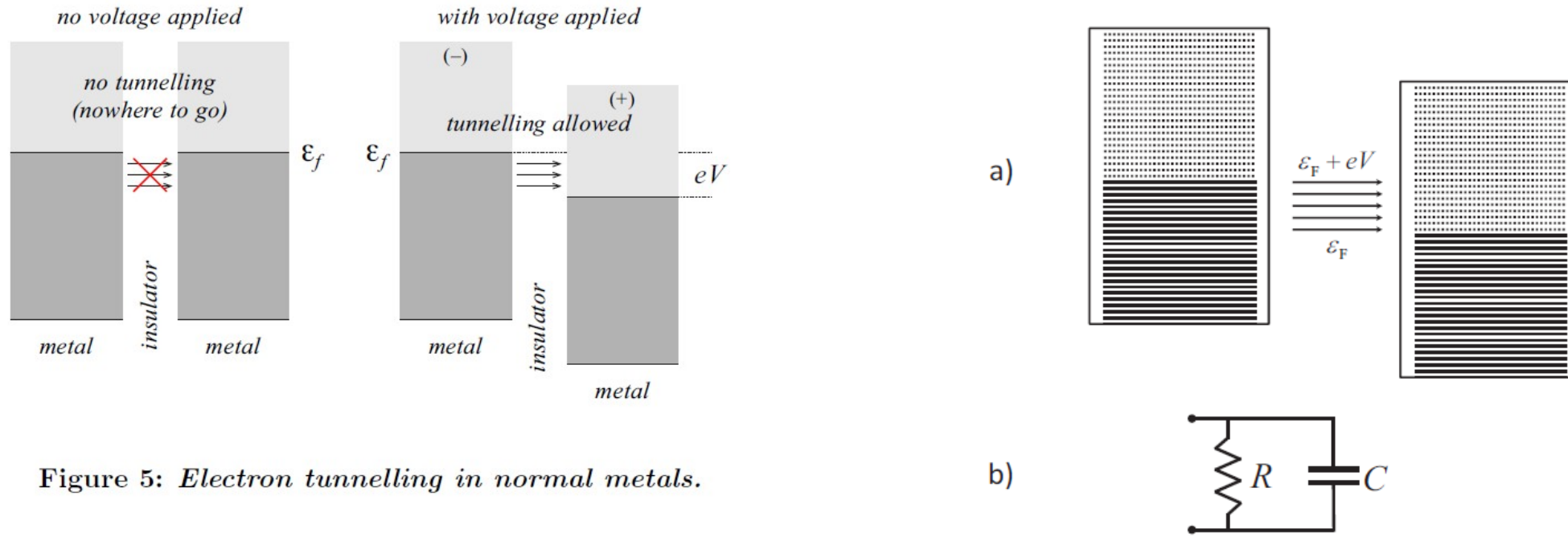


Figure 5: *Electron tunnelling in normal metals.*

Fig. 3.1 Normal metal tunnel junction with equilibrium Fermi energy ϵ_F and applied voltage bias V . (a) The single-particle spectrum in each electrode is dense and the occupied (empty) states are indicated by solid (dotted) lines. (b) Lumped-element equivalent circuit. N.B. this is an approximate equivalent circuit, only valid when the junction is voltage biased by a zero-impedance source, so that certain many-body Coulomb interaction effects are negligible.

Electronic Band Structure of a Superconductor

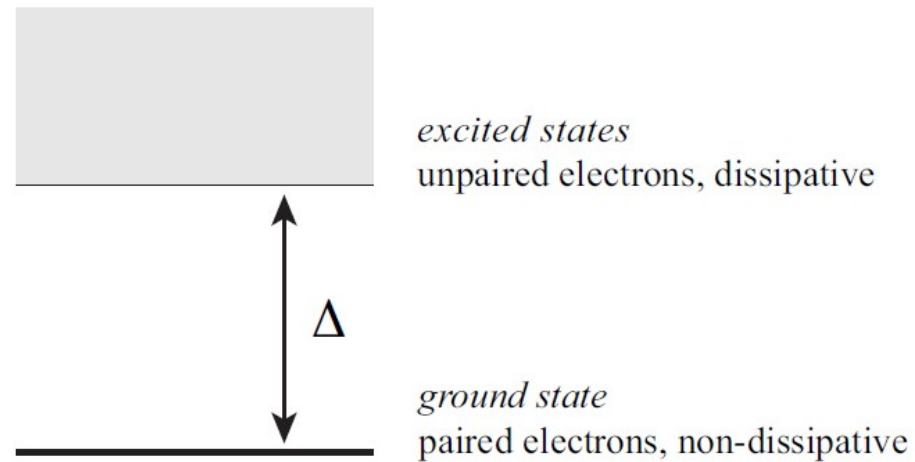


Figure 4: *Electronic band structure of a superconductor.*

Tunneling of Cooper-pair across Barrier

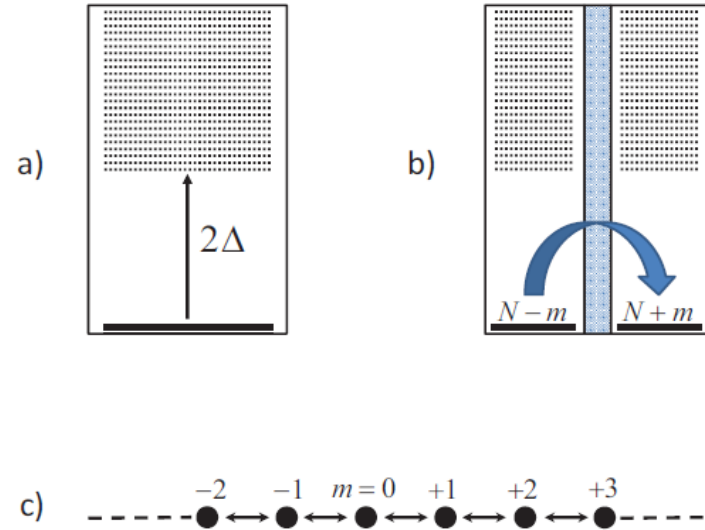


Fig. 3.2 a) Spectrum of a superconducting Cooper pair box (CPB). For the case of an even number of electrons, there is a unique non-degenerate state separated from the excited states by a gap 2Δ . b) A pair of CPB's connected by a tunnel barrier to form a Josephson junction. Ignoring the Coulomb energy, there is a large family of degenerate ground states labeled by an integer m representing the number of Cooper pairs transferred from one condensate to the other. c) 'Tight-binding' lattice along which the junction 'moves' via Josephson tunneling between 'sites' labeled by adjacent values of m .

Electron Tunneling in Superconductor

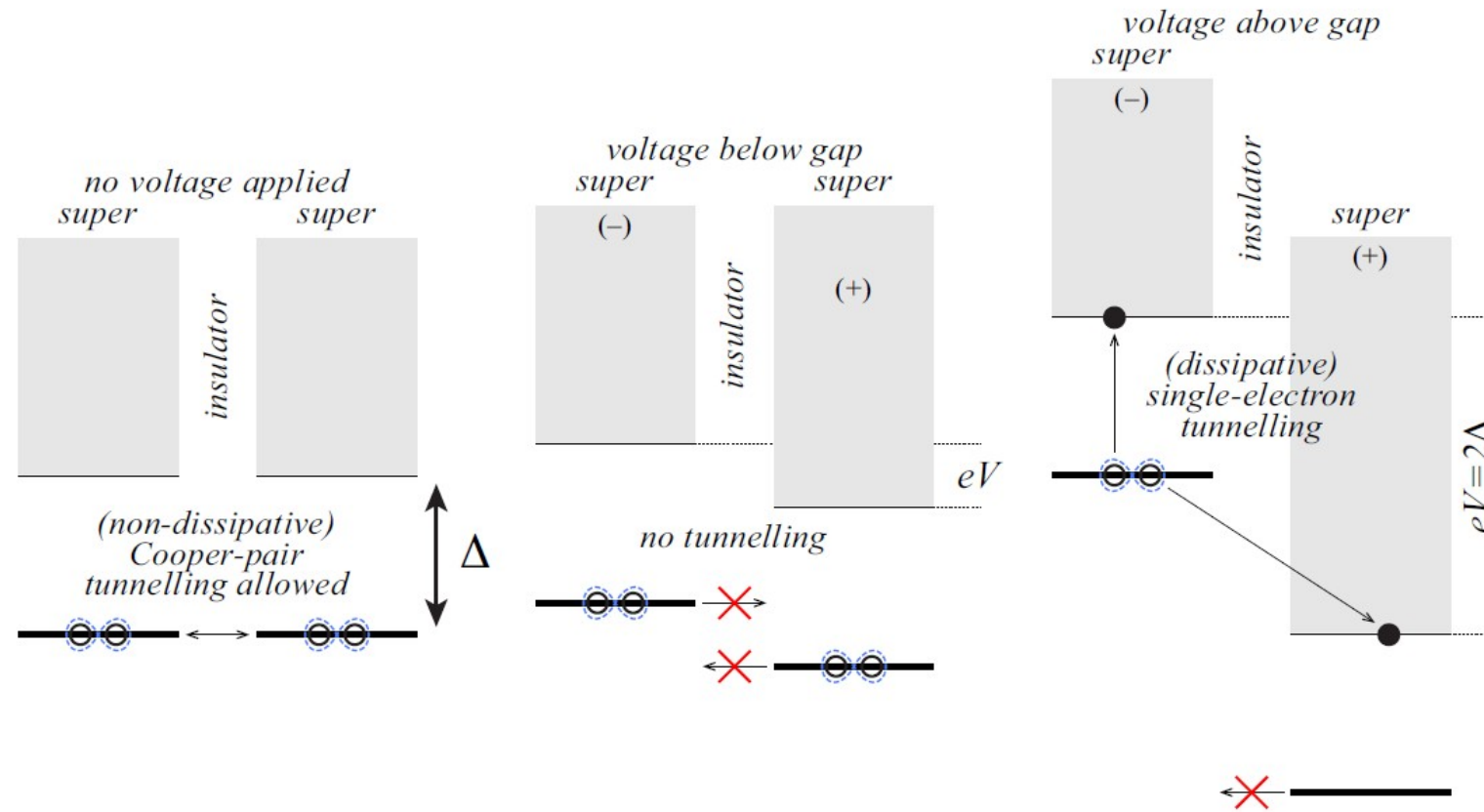


Figure 6: *Electron tunnelling in superconductors.*

Operating Temperature Range of SC-Qubit

Temperature of **1.0 K** corresponds to a frequency of approximately **21 GHz**, we will be considering **the limit of low temperature and low frequencies** relative to the gap: $k_B T, \hbar\omega \ll 2\Delta$

Josephson Junction

Two superconductors sandwiched with a layer of insulator/normal metal between

Condensed state of cooper pair in superconductors can be described in both superconductors by a **macroscopic wave function** 'whose square amplitude' is the local density of charge

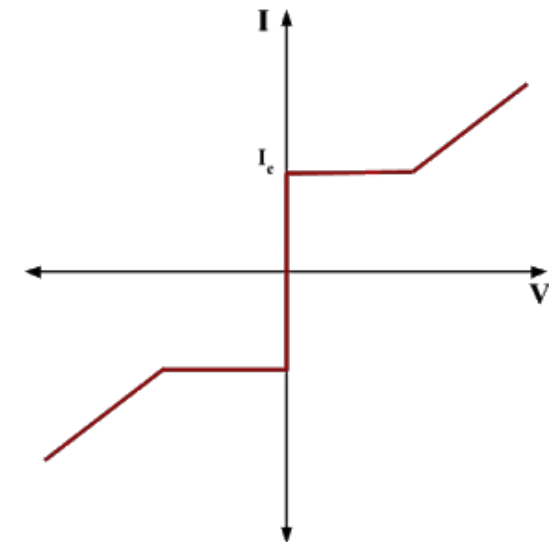
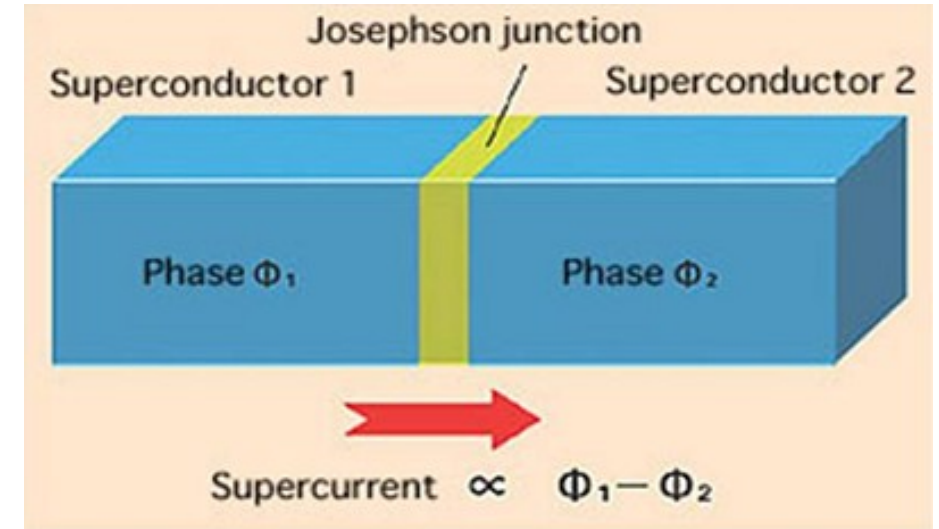
Assuming two SC are identical i.e. has same charge density

Wave functions of two superconductors

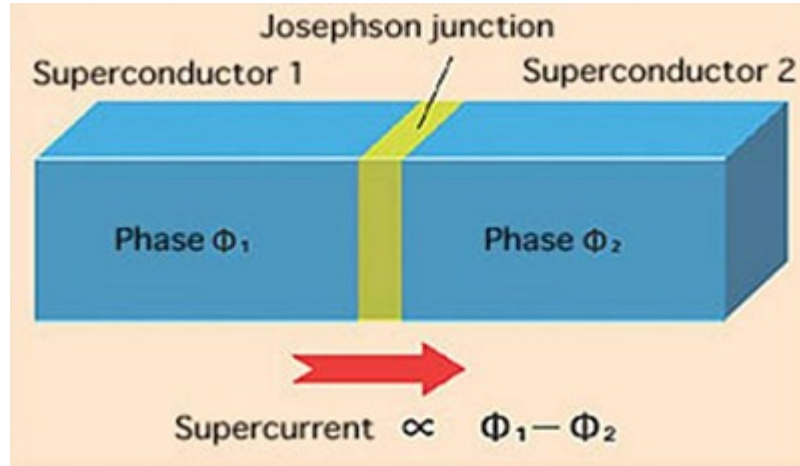
$$\psi_1(\varphi) = \sqrt{n_s} e^{i\varphi_1}$$

$$\psi_2(\varphi) = \sqrt{n_s} e^{i\varphi_2}$$

Typical I vs V sketch for a Josephson junction



Josephson Junction



Wave functions of two superconductors

$$\psi_1(\varphi) = \sqrt{n_s} e^{i\varphi_1}$$

$$\psi_2(\varphi) = \sqrt{n_s} e^{i\varphi_2}$$

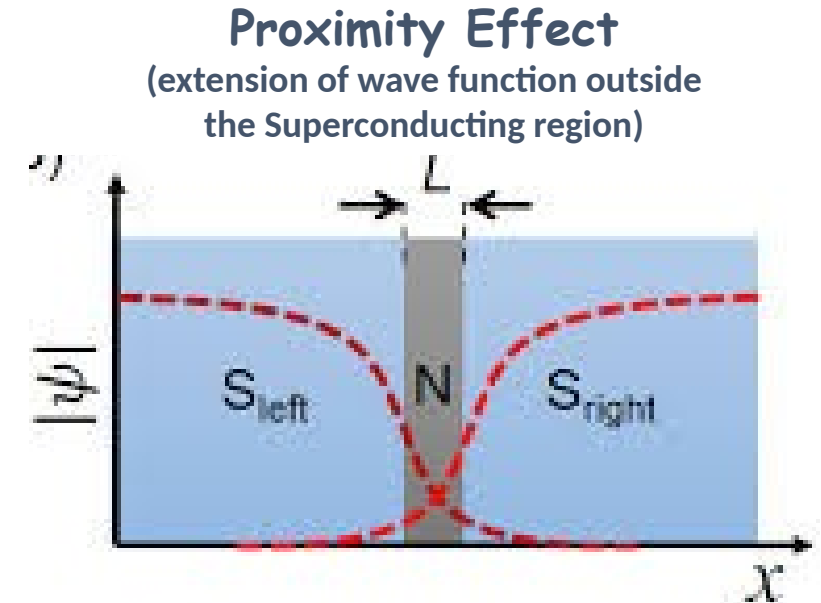
Schrodinger wave equation for above junction

$$i\hbar \frac{\partial}{\partial t} \psi_1 = \hat{H}_1 \psi_1 + T \psi_2 \quad \dots\dots\dots(1)$$

$$i\hbar \frac{\partial}{\partial t} \psi_2 = \hat{H}_2 \psi_2 + T \psi_1 \quad \dots\dots\dots(2)$$

Where T is coupling constant

Which is characteristic of tunnel barrier



Proximity Josephson coupling

Proximity allows **coherence tunneling** of coher pair gap between SC

Josephson Junction

Now subst. following in Eq. (1) and (2) and applying biasing voltage V

$$H_2\psi_2 = E_2\psi_2$$

$$H_1\psi_1 = E_1\psi_1$$



$$i\hbar \frac{\partial}{\partial t} \psi_1 = \frac{qV}{2} \psi_1 + T\psi_2 \quad \dots\dots\dots(3)$$

$$i\hbar \frac{\partial}{\partial t} \psi_2 = \frac{qV}{2} \psi_2 + T\psi_1 \quad \dots\dots\dots(4)$$

Now subst. following in Eq. (3) and (4)

$$\psi_1 = \sqrt{n_s} e^{i\varphi_1}$$

$$\psi_2 = \sqrt{n_s} e^{i\varphi_2}$$



$$\dot{n}_s = \frac{2T}{\hbar} n_s \sin(\varphi_2 - \varphi_1) \equiv I_J \quad \dots\dots\dots(5)$$

DC Josephson effect

$$\hbar(\dot{\varphi}_2 - \dot{\varphi}_1) = qV \quad \dots\dots\dots(6)$$

AC Josephson effect

The phase difference depends upon the applied voltage

$$\varphi_2 - \varphi_1 = \frac{1}{\hbar} qVt + \Delta\varphi_0$$

$\Delta\varphi_0$ is arbitrary initial phase



$$\dot{n}_s = \frac{2T}{\hbar} n_s \sin\left(\frac{1}{\hbar} 2eVt + \Delta\varphi_0\right) \equiv I_J$$

$$I_J = \frac{2T}{\hbar} n_s \sin(\omega_J t + \Delta\varphi_0)$$

$$\omega_J = \frac{1}{\hbar} 2eV$$

Josephson Frequency

Josephson Effect (equations)

$$I_J = \frac{2T}{\hbar} n_s \sin(\omega_J t + \Delta\varphi_0)$$



$$I = I_c \sin(\omega_J + \Delta\varphi_0)$$

Where $\omega_J = \frac{1}{\hbar} 2eV$

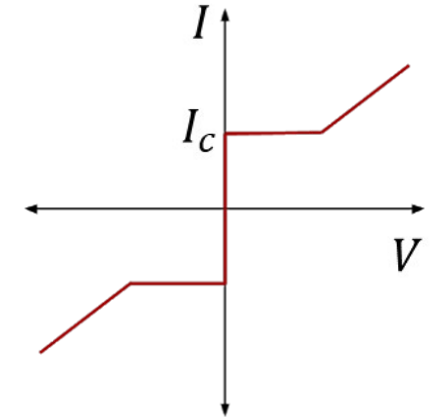
Josephson current in the presence of voltage across the junction

Where I_c is called critical current

$$I = I_c \sin(\Delta\varphi) \dots\dots\dots(1)$$

Total phase difference
 $\Delta\varphi$ – is generally not constant !

$$V = \frac{\hbar}{2e} \frac{d\Delta\varphi}{dt} \dots\dots\dots(2)$$



Equation (1) and (2) are known as **first and second Josephson equations**, respectively

From (1) $\frac{dI}{dt} = I_c \cos\Delta\varphi \frac{d\Delta\varphi}{dt} \dots\dots\dots(3)$

Substituting (3) in (2)  $V = \frac{\hbar}{2e} \frac{1}{I_c \cos\Delta\varphi} \frac{dI}{dt} \dots\dots\dots(4)$

Josephson Effect (equations)

$$V = \frac{\hbar}{2e} \frac{I}{I_c \cos \Delta \varphi} \frac{dI}{dt} \dots\dots\dots(4)$$

We know $V = L \frac{dI}{dt} \dots\dots\dots(5)$ Definition of Inductance

From (4) and (5) $L = \frac{\hbar}{2e} \frac{1}{I_c \cos \Delta \varphi} \equiv L_J$ Called **Josephson inductance**

$$L = \frac{\Phi_0}{2\pi} \frac{1}{I_c \cos \Delta \varphi} \equiv L_J$$

In terms of Flux Quanta

Josephson inductance L_J , depends upon phase difference $\Delta \varphi$.

L_J is not constant: make the LC circuit (with Josephson junction) nonlinear

AC Josephson Effect

Consider a constant (DC) voltage driving the Josephson junction

$$V = \text{constant}$$

$$V = \frac{\hbar}{2e} \frac{d\Delta\varphi}{dt} = \text{constant} \quad \Rightarrow \quad \Delta\varphi = \frac{2eV}{\hbar} t$$

Therefore from First Josephson

$$I = I_c \sin(\Delta\varphi)$$

$$\Rightarrow I = I_c \sin\left(\frac{2eV}{\hbar} t\right)$$

$$I = I_c \sin(\omega_J t) \quad \text{Where } \omega_J = \frac{2eV}{\hbar}$$

Known as **Josephson Frequency**

Current I varies sinusoidally with frequency ω_J - controlled by the voltage across the Josephson junction

DC Josephson Effect

When current through Josephson junction is constant:

$$I = \text{constant}$$

$$I < I_c \text{ always}$$

Therefore from First Josephson eq. $I = I_c \sin(\Delta\varphi)$

$$\Rightarrow \Delta\varphi = \sin^{-1}\left(\frac{I}{I_c}\right)$$

Therefore Phase difference $\Delta\varphi$, across the Josephson junction can be calculated by measuring the current across the junction

Remember for **DC Josephson Junction**

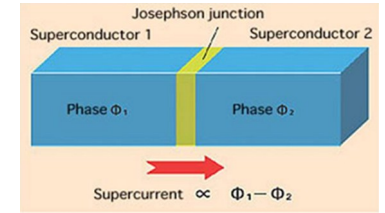
$$\frac{d\Delta\varphi}{dt} = 0$$

Therefore from Second Josephson eq.

$$V = \frac{\hbar}{2e} \frac{d\Delta\varphi}{dt}$$

$$\Rightarrow V = 0$$

Josephson Energy



$$\dot{n}_s = \frac{2T}{\hbar} n_s \sin(\varphi_2 - \varphi_1) \equiv I_J$$

$$I_J = I_c \sin \Delta\varphi$$

$$E = \int I(t) V(t) dt$$

$$E = \int I_c \sin \Delta\varphi \frac{\Phi_0}{2\pi} \frac{d\Delta\varphi}{dt} dt = \frac{I_c \Phi_0}{2\pi} \int \sin \Delta\varphi \frac{d\Delta\varphi}{dt} dt$$

$$E = E_J \int_{-\infty}^t \sin \Delta\varphi \frac{d\Delta\varphi}{dt} dt$$

Where $E_J = \frac{I_c \Phi_0}{2\pi}$

E_J is cooper-pair energy tunneling across the SIS junction

$$E = E_J (1 - \cos(\Delta\varphi))$$

Harmonic to Anharmonic Oscillator

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \hat{\phi}^2$$

$$\omega_{LC} = \omega = \frac{1}{\sqrt{LC}}$$

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega^2 \hat{\phi}^2$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$



To diagonalize the Hamiltonian we introduced and expressed the $\hat{Q}$, $\hat{\phi}$, and $\hat{H}$ in-terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2)$$

Now the let us write Hamiltonian of oscillator with JJ

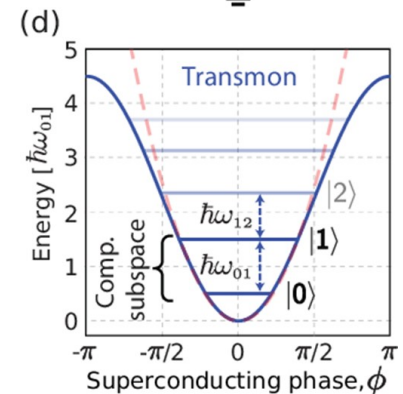
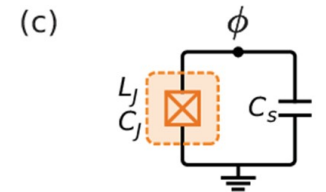
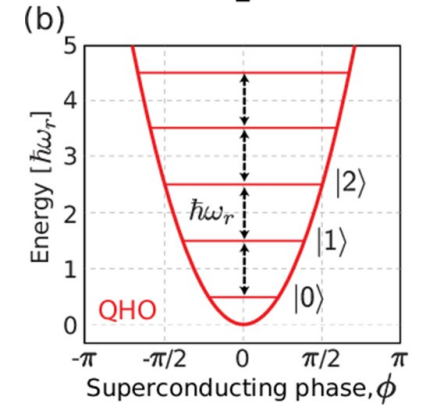
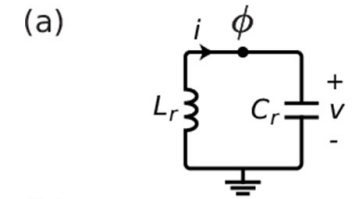
$$\hat{H} = \frac{\hat{Q}^2}{2C} + E_J (1 - \cos(\Delta\hat{\phi}))$$

Q (charge) can be replaces by $2ne$
Because in SC current is because of cooper-pairs

$$\hat{H} = \frac{(2e)^2}{2C} \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi}))$$

Instead of $\hat{Q}$ now we have $\hat{n}$ (charge number operator)

$$\hat{H} = 4 \frac{e^2}{2C} \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi})) \quad \longrightarrow \quad \hat{H} = 4E_C \hat{n}^2 + E_J (1 - \cos(\Delta\hat{\phi}))$$



Harmonic to Anharmonic Oscillator

Hamiltonian of oscillator with JJ

$$\hat{H} = 4E_c \hat{n}^2 + E_J (1 - \cos(\Delta \hat{\phi}))$$

Using cosine series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\hat{H} = 4E_c \hat{n}^2 + E_J \left\{ \frac{(\Delta \hat{\phi})^2}{2!} - \frac{(\Delta \hat{\phi})^4}{4!} + \frac{(\Delta \hat{\phi})^6}{6!} - \frac{(\Delta \hat{\phi})^8}{8!} + \dots \right\}$$

$$\hat{H} = \underbrace{4E_c \hat{n}^2 + \frac{E_J}{2} (\Delta \hat{\phi})^2}_{\text{LHO}} - \underbrace{\frac{E_J}{24} (\Delta \hat{\phi})^4 - \dots}_{\text{Non-linear effects are from these terms}}$$

LHO

Non-linear effects are from these terms

Using convention process we can express above Hamiltonian in terms of ladder operators $\hat{b}$ and $\hat{b}^\dagger$ (substituting $\hbar = 1$)

$$\hat{H} = \omega_J \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) - \frac{E_c}{12} (\hat{b}^\dagger + \hat{b})^4$$

Comparing this Hamiltonian with that of LHO

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \hat{\phi}^2$$



$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_{LC}^2 \hat{\phi}^2$$

$$4E_c \leftrightarrow \frac{1}{2C}$$

$$\frac{E_J}{2} \leftrightarrow \frac{1}{2L}$$

$$\omega_J = \sqrt{8E_J E_c} \leftrightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}$$

$$\hat{H} = \omega_J \hat{b}^\dagger \hat{b} - \frac{E_c}{12} (\hat{b}^\dagger + \hat{b})^4$$

$$\hat{H} = \omega_J \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + 2 \hat{b}^\dagger \hat{b})$$

$$\hat{H} = (\omega_J - E_c) \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})$$

$$\hat{H} = \omega_{01} \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})$$

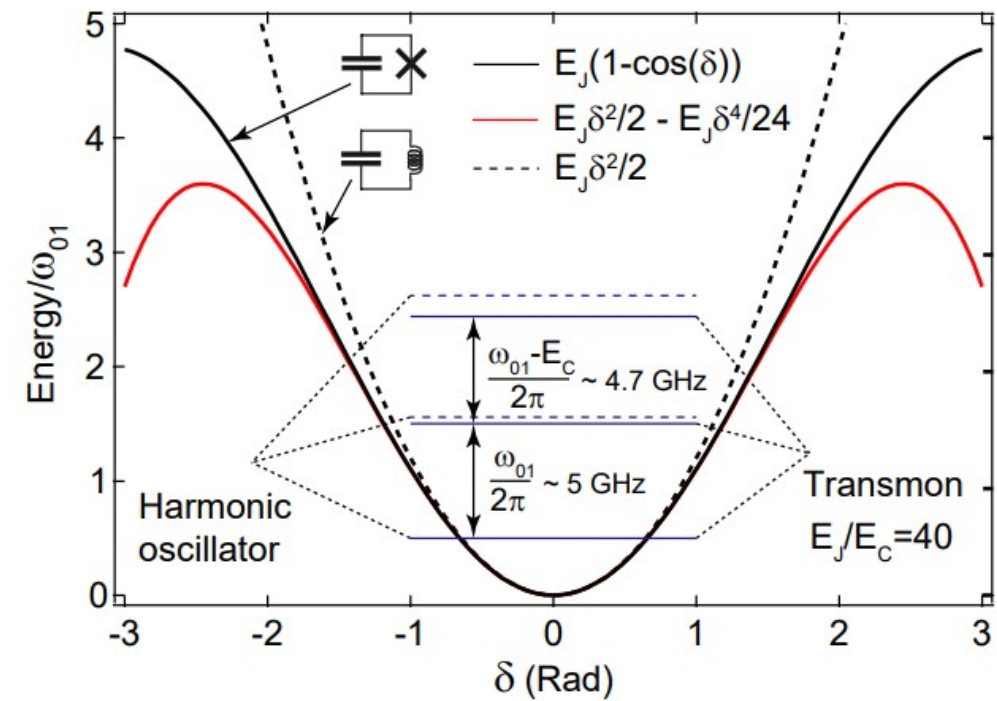
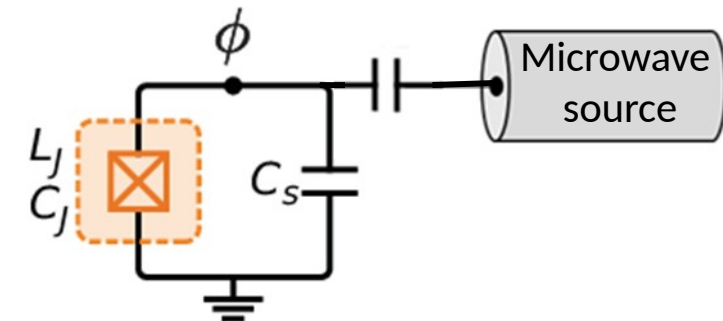


Figure 2.9: **Transmon energy levels:** A typical transmon ($E_J/E_C = 40$) potential (Eq (2.2.8) in solid black curve) in comparison with a nonlinear oscillator (Eq. (2.2.9) in the solid red curve) and a harmonic oscillator (dashed parabola). The first three energy levels are also depicted for the transmon (nonlinear oscillator) in comparison to the harmonic oscillator. The typical values for transition energies/frequencies are shown. Note $E_{01} = \hbar\omega_{01}$ and we set $\hbar = 1$.

You can change the state of qubit by applying suitable MW source

$$\hat{H} = 4E_c \hat{n}^2 + E_J(1 - \cos(\Delta\hat{\phi}))$$

$$\hat{H} = 4E_c (\hat{n} - n_g)^2 + E_J(1 - \cos(\Delta\hat{\phi}))$$



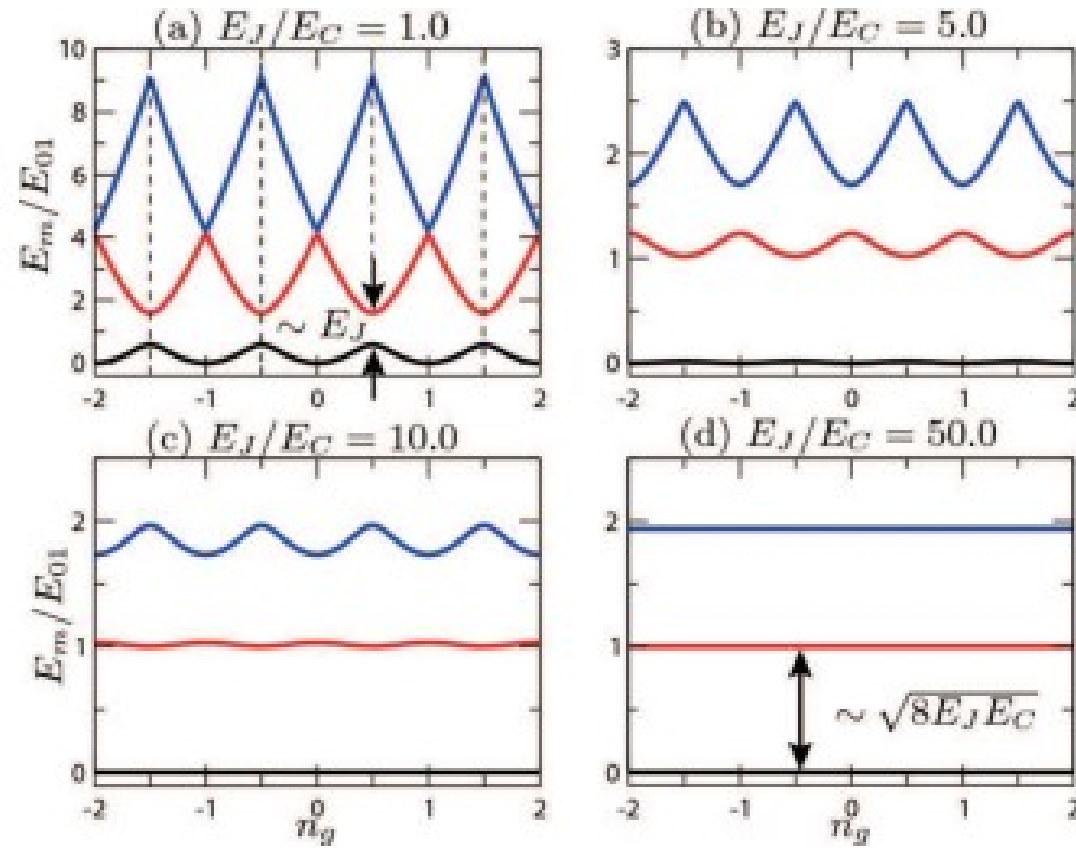
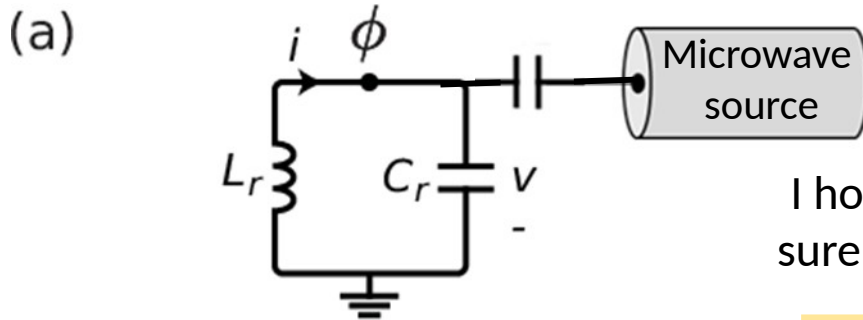


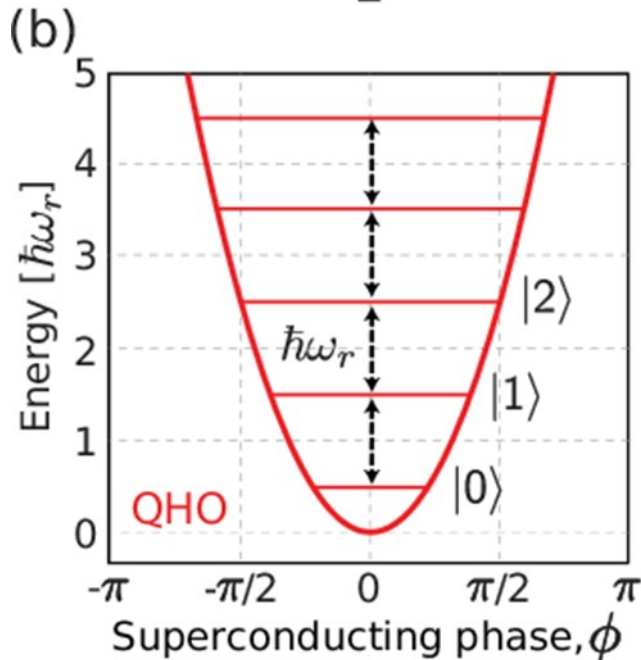
Fig. 4.6 Energy spectrum of the Cooper pair box as a function of offset charge for different values of the dimensionless ratio of Josephson energy to charging energy. The exponential decrease in the charge dispersion is clearly seen. From (Koch *et al.*, 2007).

Coherent States of Harmonic Oscillator

Assume initially both the oscillators (**harmonic** and **anharmonic**) are in ground state



I hope you know how to keep/make sure that they remain in ground state



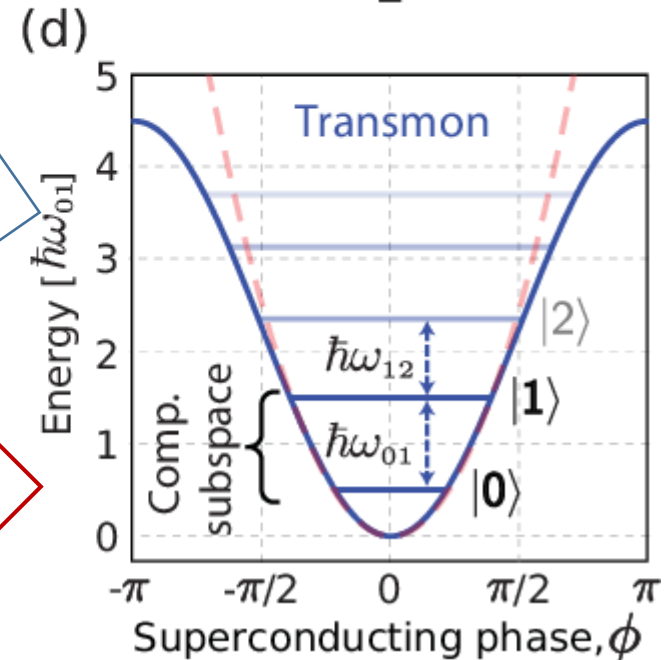
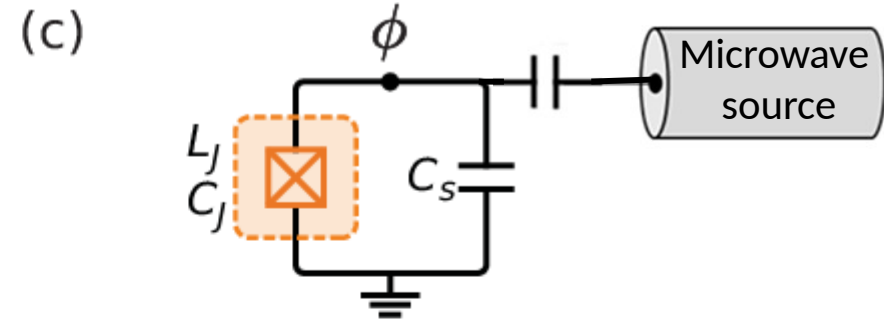
Now couple the circuits to MW sources of suitable frequency

After MW coupling the oscillators are in 'Coherent state'

How many sates are occupied

How many sates are occupied

Which of the oscillator is a 'good' qubit?



Coherent state concept is very important to built a qubit using LC/JJ-based circuit

Coherent States of Harmonic Oscillator

Assume initially both the oscillators (**harmonic** and **anharmonic**) are in ground state

Now couple the circuits to MW sources of suitable frequency

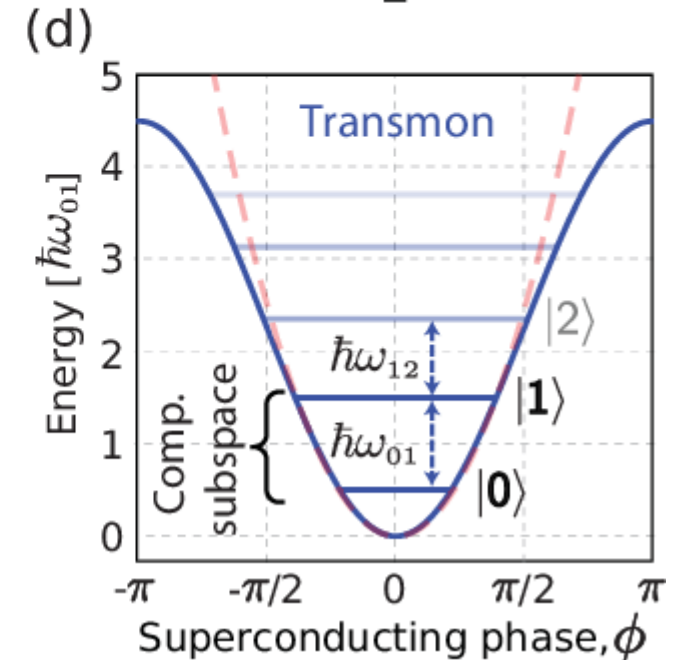
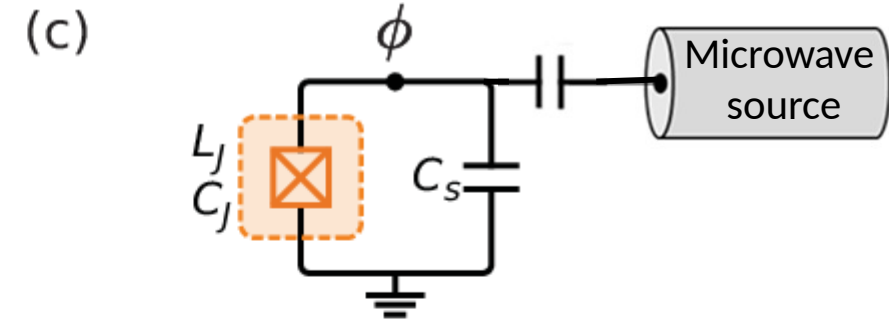
After MW coupling the oscillators are in 'Coherent state'

How much time is required to create a coherent state?

It depends upon the coupling strength (/dipole moment)

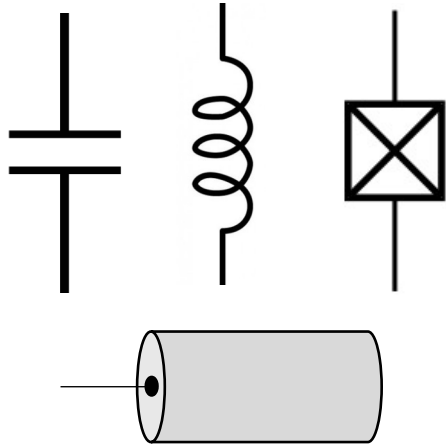
For circuit qubit: $4 - 20 \text{ ns}$ **Very short pulse is enough**

For ion (because of very small dipole): *in the rang of μs*

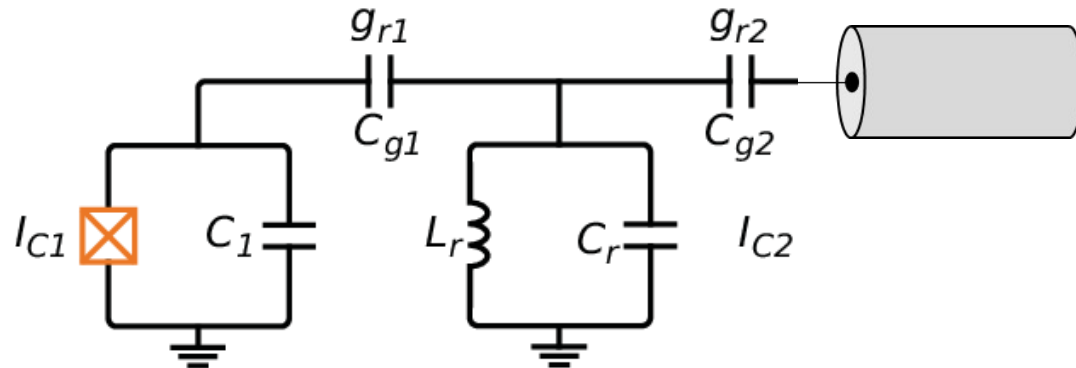


Strong coupling ability of circuit qubit is not always good – **responsible for unwanted coupling as well**

Circuit QED ingredients



Lumped circuit element for circuit QED

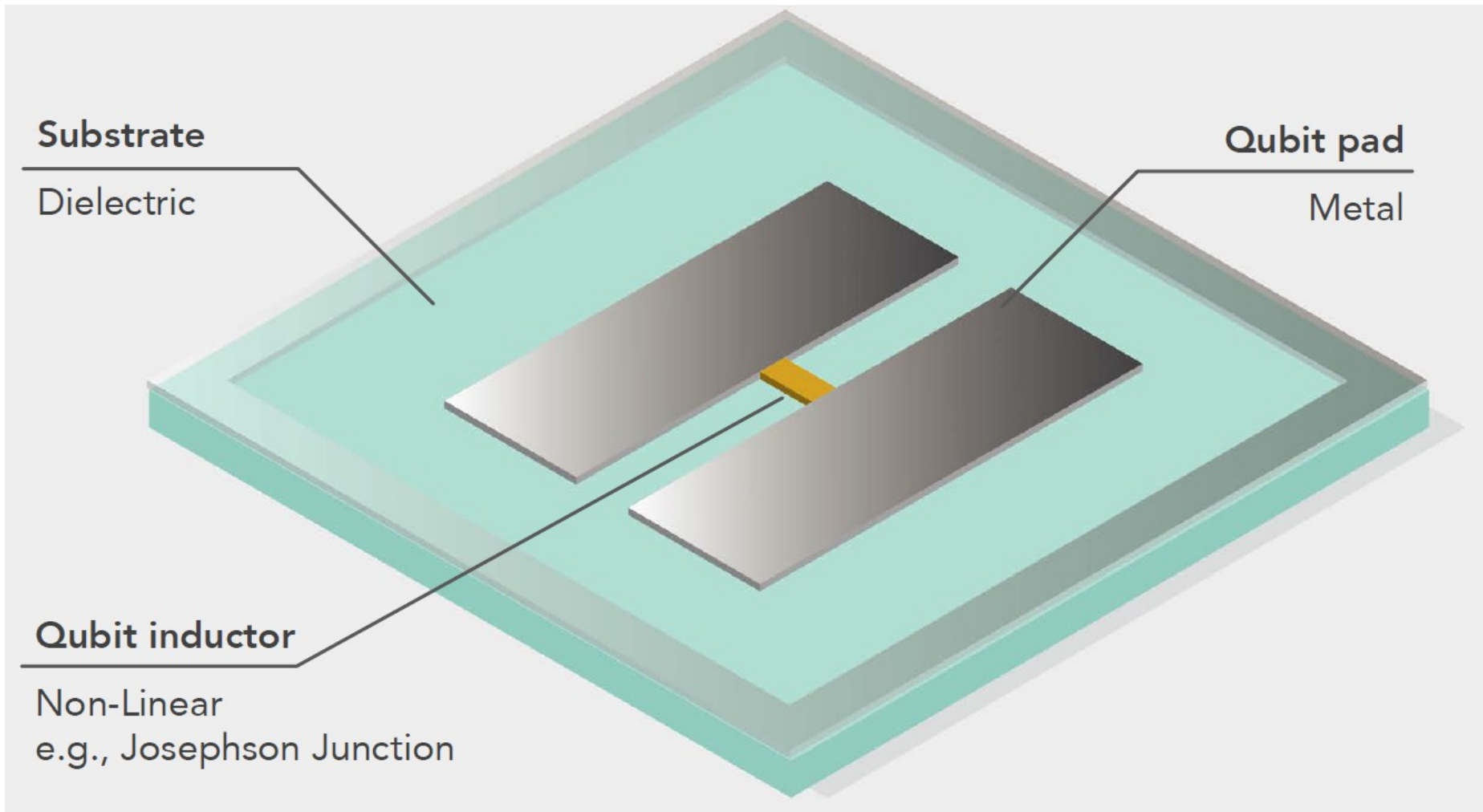


Capacitive coupling

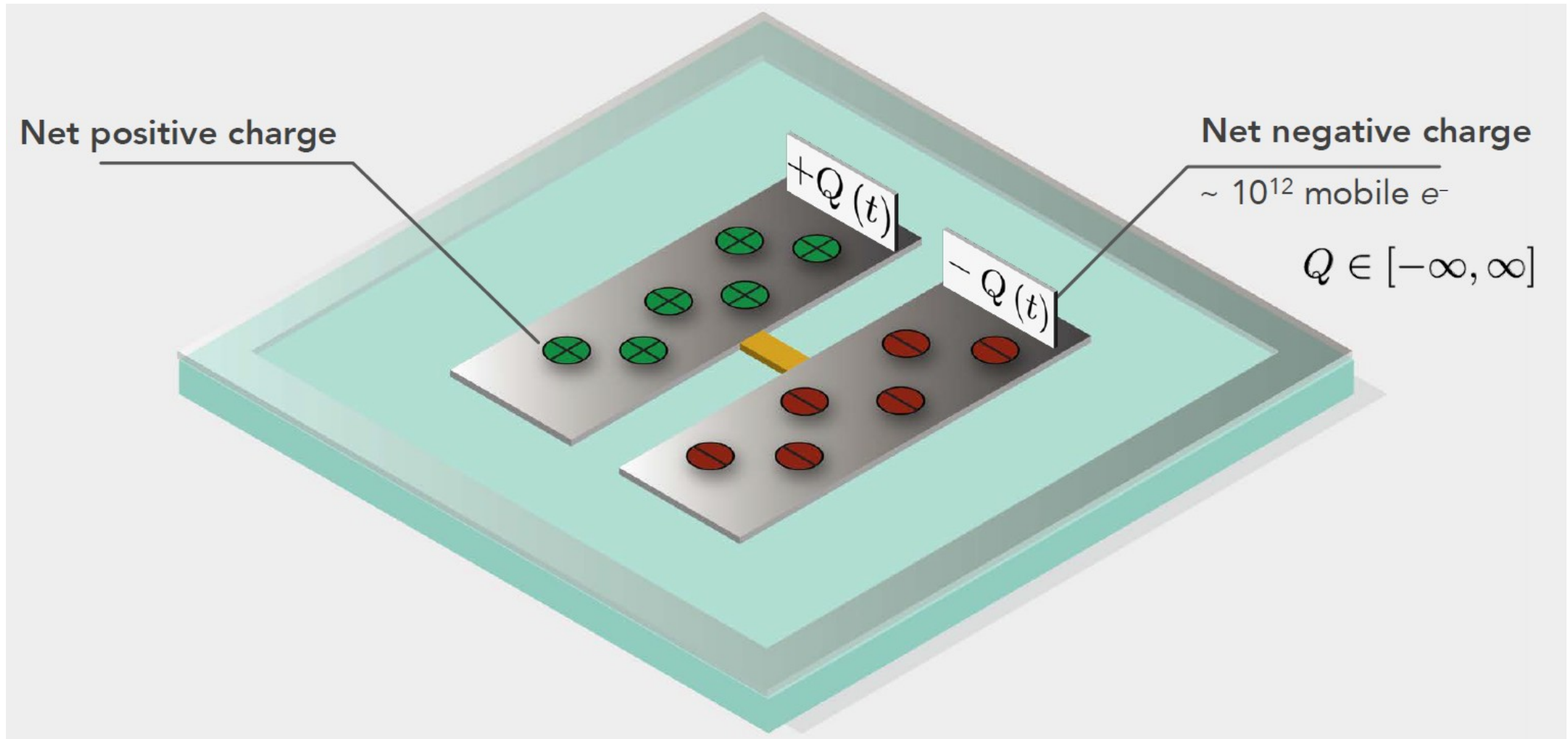
JJ-based circuits including different types of Qubits

Circuit	Split Cooper-pair box	Quantronium	Transmon
Features	Tunable Josephson energy	Charge-flux qubit	Charge noise reduction
Circuit	Xmon	Gmon	Fluxonium
Features	Connectivity	Tunable coupling	Charge noise reduction
Circuit	3-junction flux qubit	C-shunt flux qubit	Tunable-gap flux qubit
Features	Reduction of loop size	Charge noise reduction	Tunability

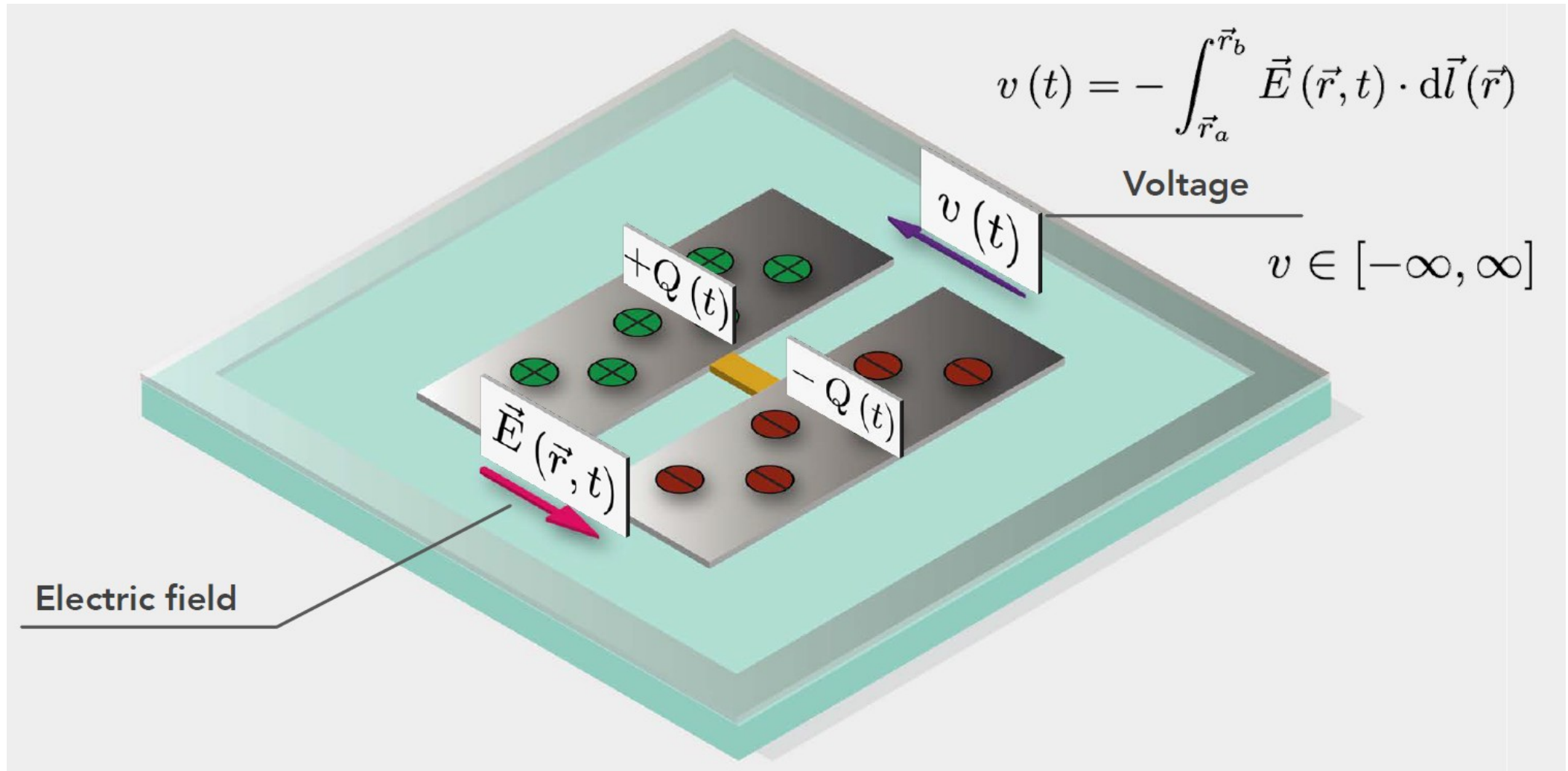
Transmon Qubit



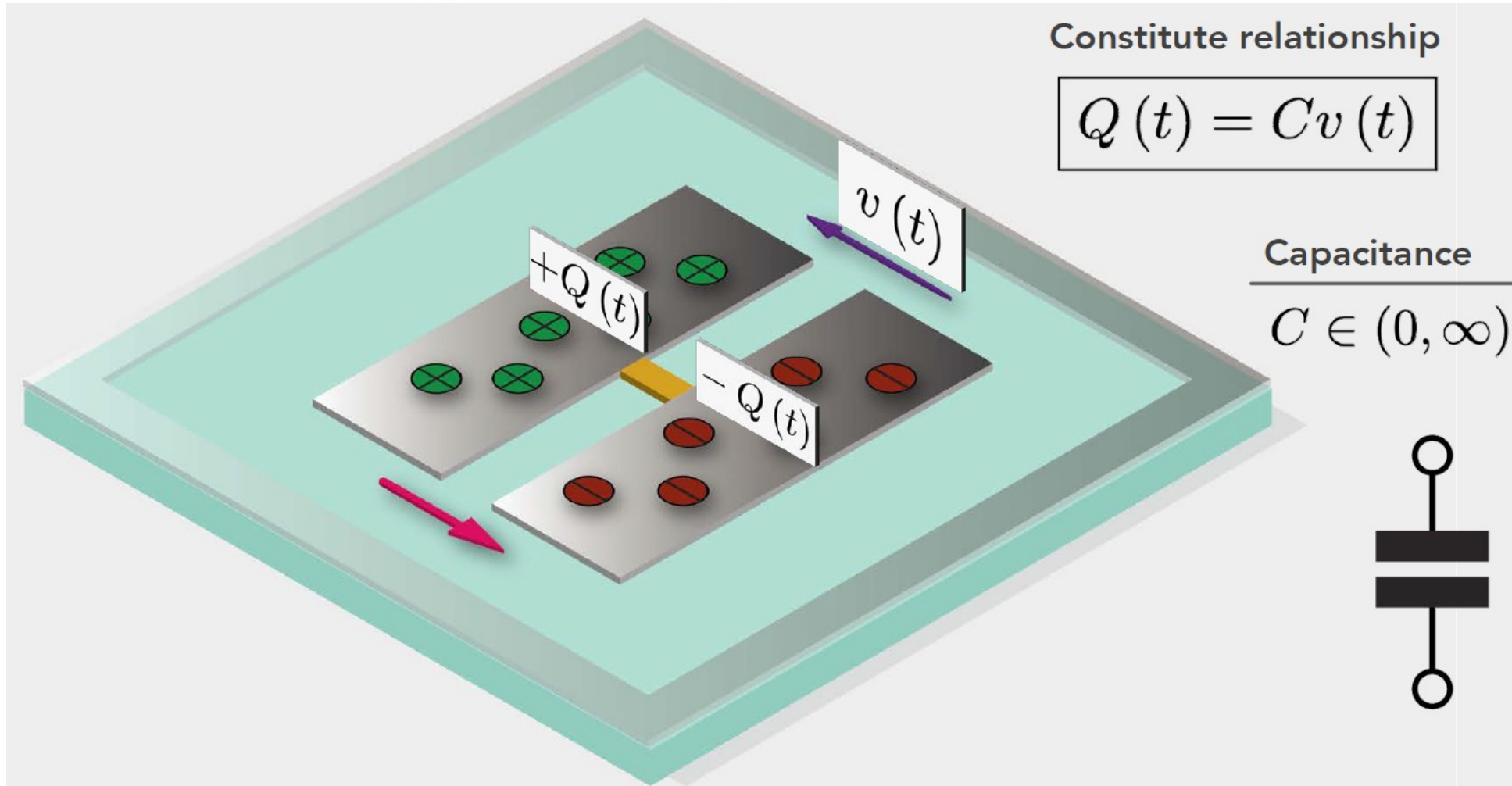
Transmon Qubit: Charge



Electric Field and Voltage



Charge and Capacitance



Charge and Current

Conservation of charge

Universal relationship

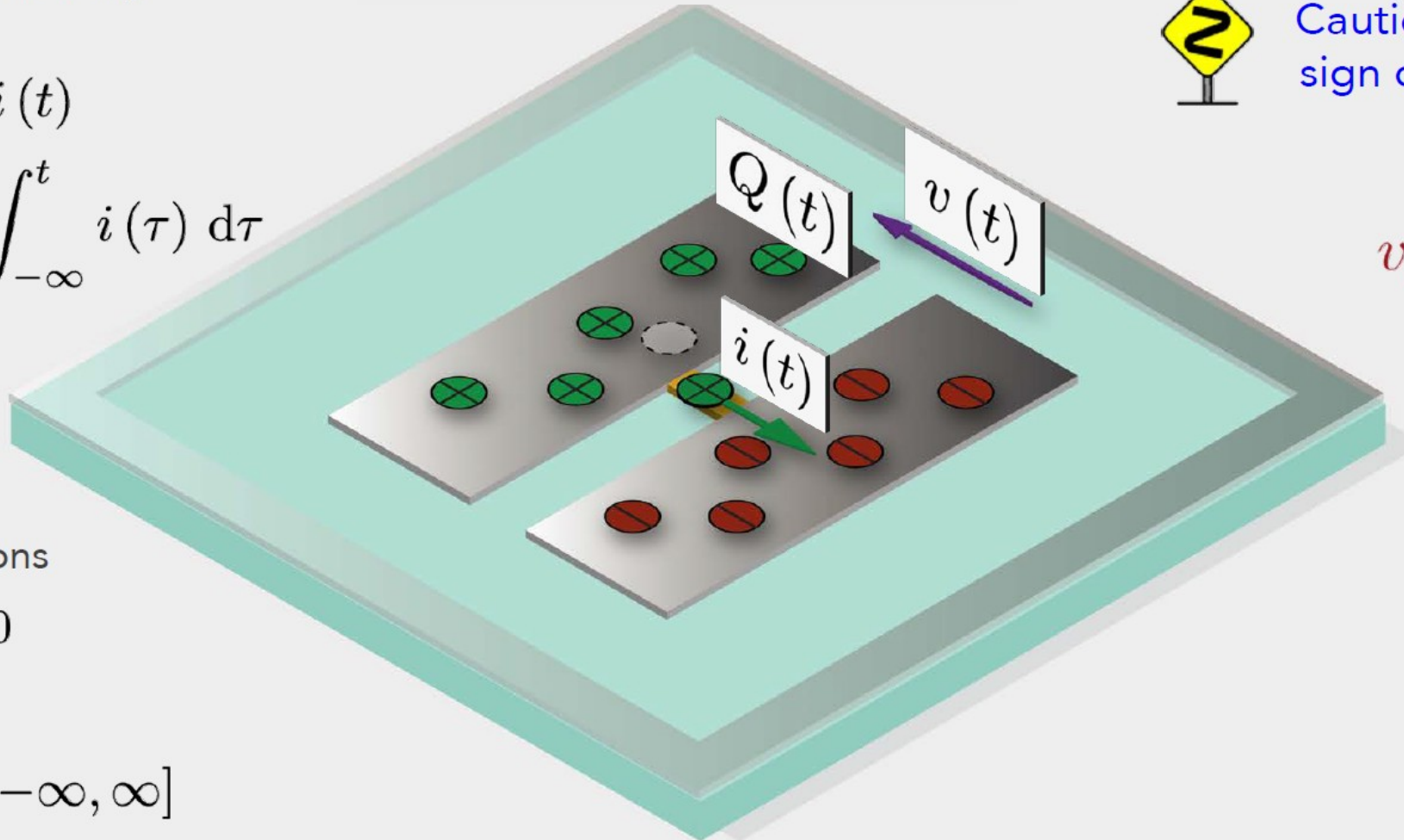
$$\frac{d}{dt}Q(t) = i(t)$$

$$Q(t) = \int_{-\infty}^t i(\tau) d\tau$$

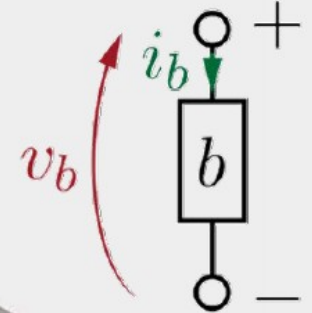
Initial conditions

$$Q(-\infty) = 0$$

$$i \in [-\infty, \infty]$$



Caution: Passive sign convention



Magnetic Flux and Inductance

Faraday's law of induction

Universal relationship

$$\Phi(t) = \int_{-\infty}^t v(\tau) d\tau$$

$$\frac{d}{dt}\Phi(t) = v(t)$$

Initial conditions

$$\Phi(-\infty) = 0$$

Magnetic flux

For kinetic inductors,
~98% of qubit inductive energy is not
in stored magnetic fields, but in
kinetic inductance

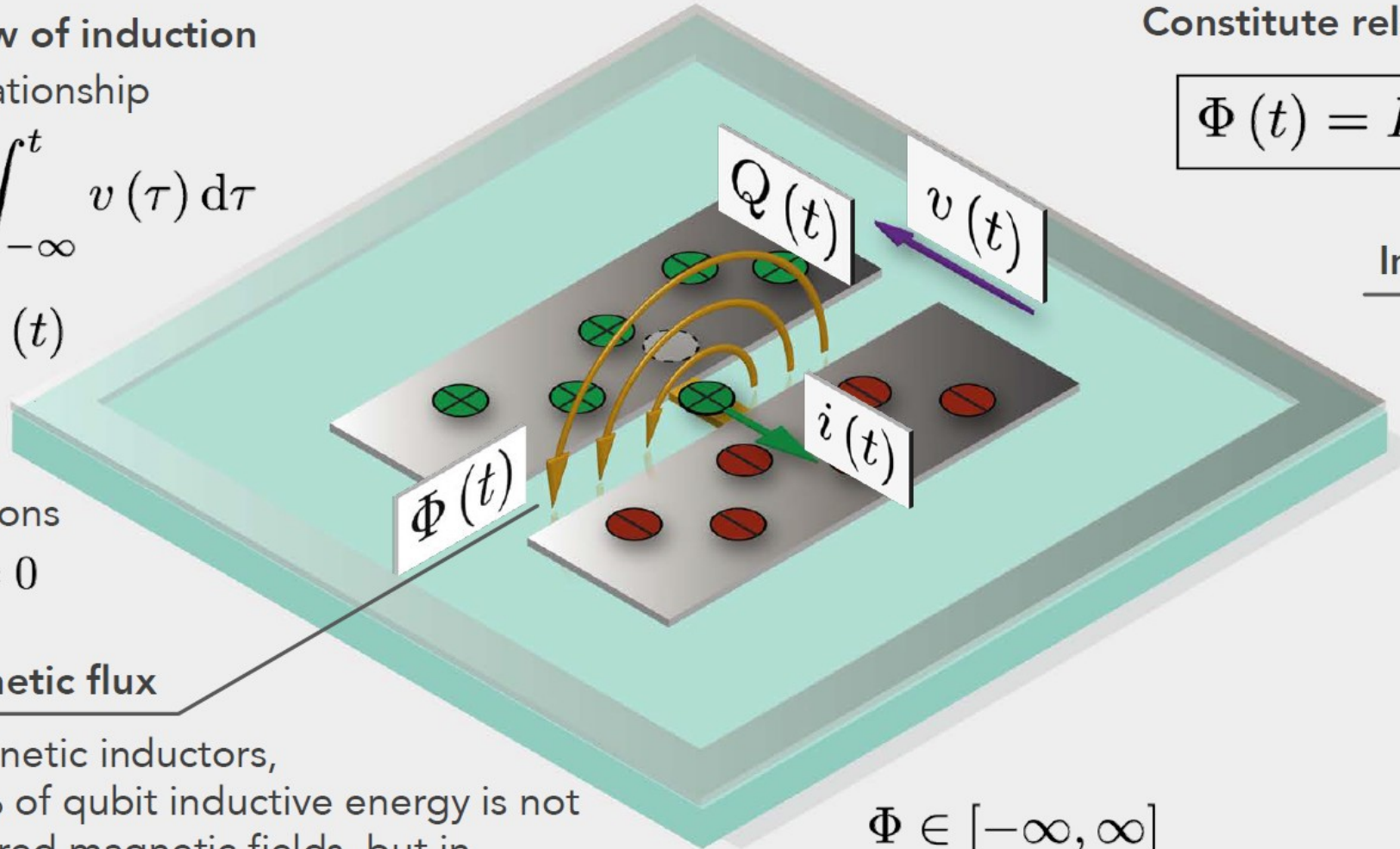
Constitute relationship

$$\Phi(t) = Li(t)$$

Inductance

$$L \in (0, \infty)$$

$$\Phi \in [-\infty, \infty]$$



Power and Energy

Universal

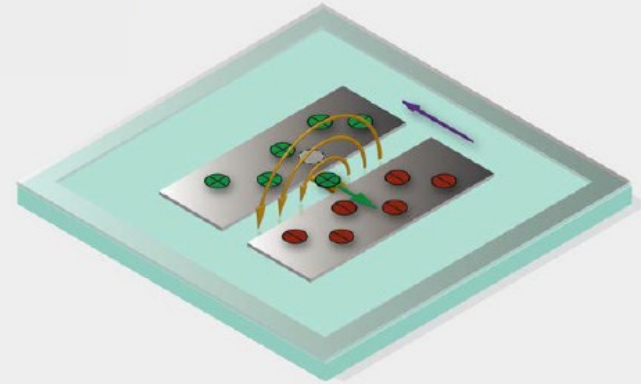
$$\frac{d}{dt}\mathcal{E}(t) = p(t) \equiv v(t)i(t)$$



Energy stored in
(delivered to)
component

Instantaneous
power flowing
to component

$$\mathcal{E}(t) = \int_{-\infty}^t p(\tau) d\tau$$



$$\mathcal{E}_{\text{cap}}(\dot{\Phi}) = \frac{1}{2}C\dot{\Phi}^2$$

$$\mathcal{E}_{\text{cap}}(Q) = \frac{Q^2}{2C}$$

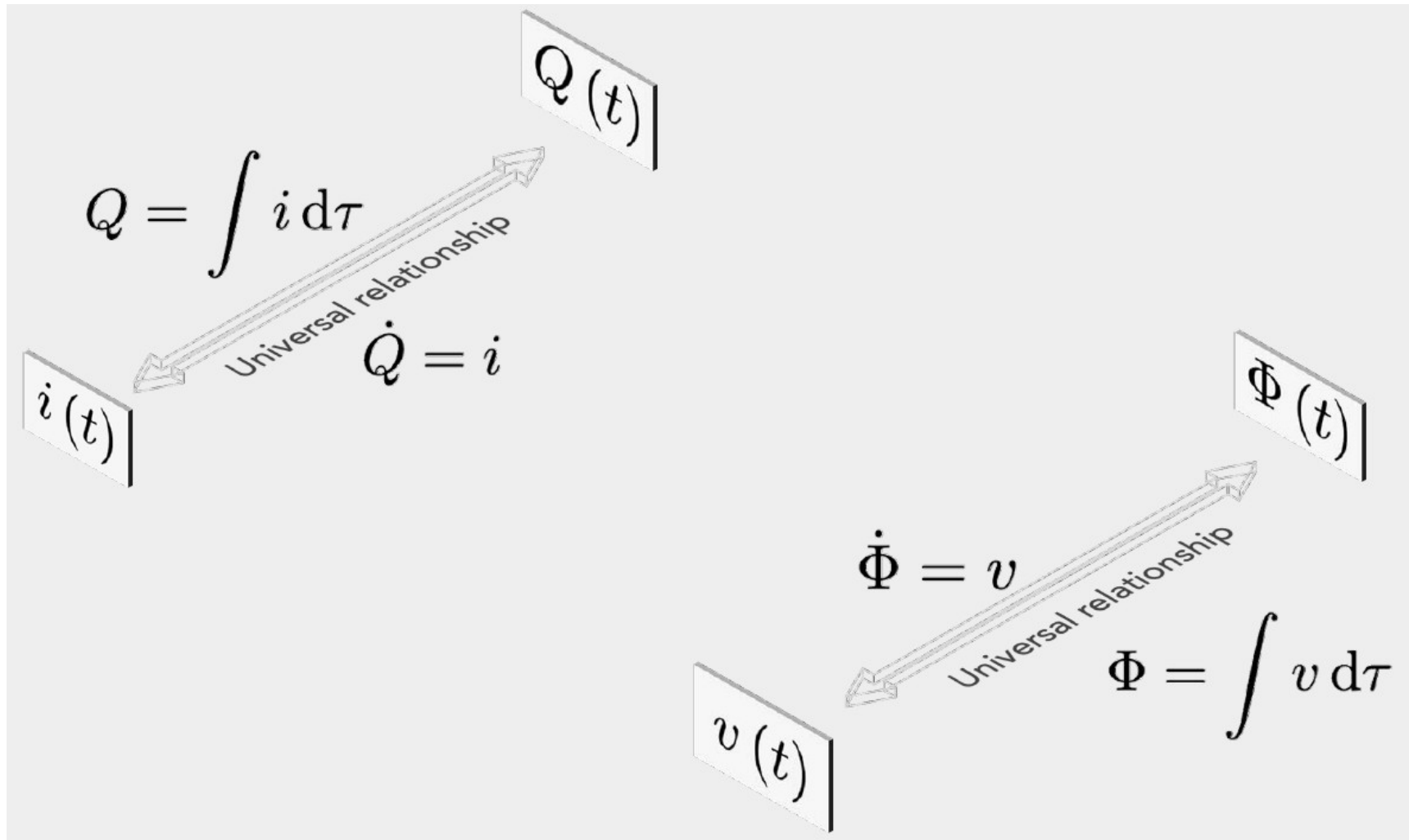


$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

$$\mathcal{E}_{\text{ind}}(\dot{Q}) = \frac{1}{2}L\dot{Q}^2$$

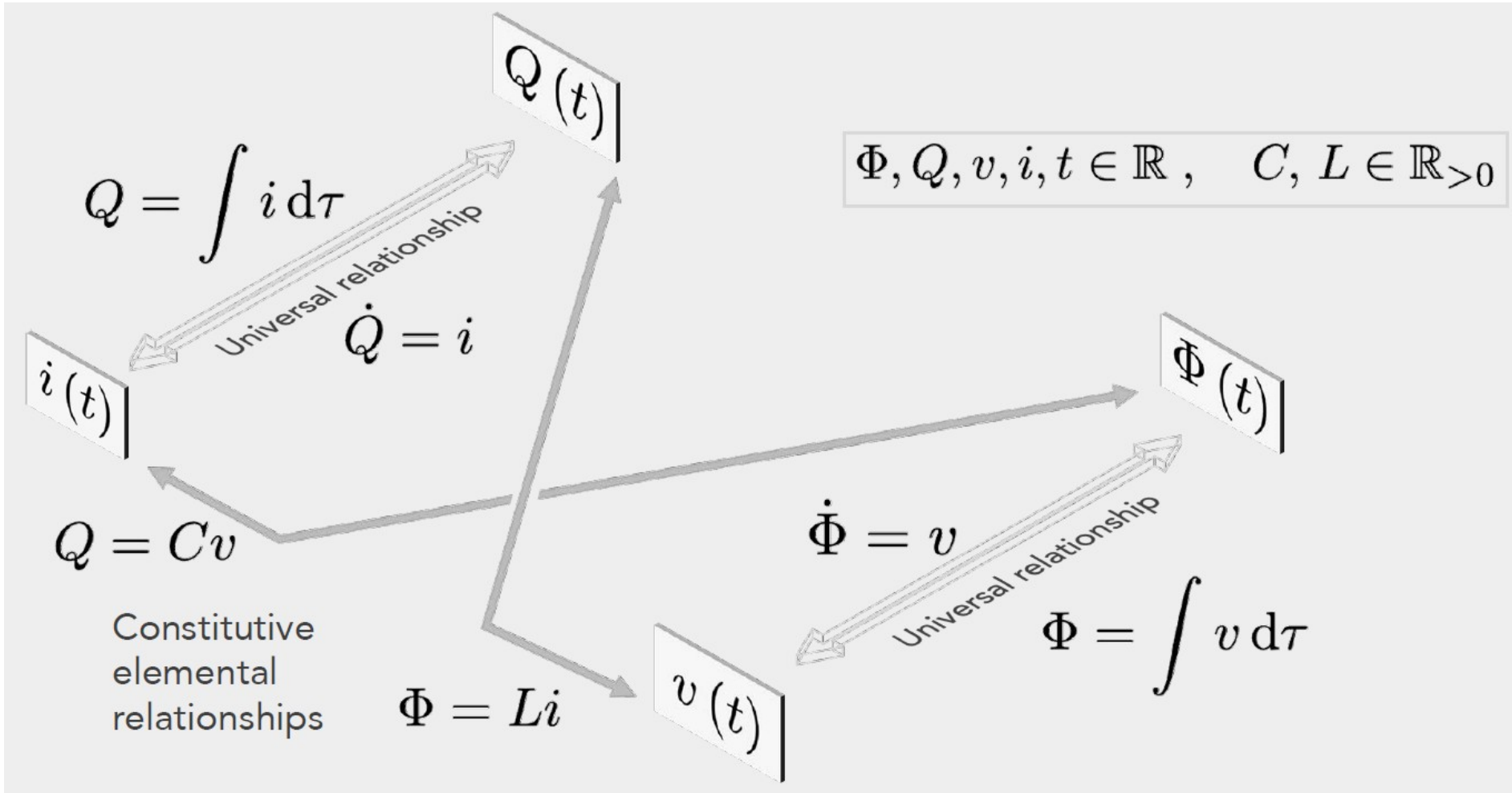
Four Circuit parameters

Charge and current, Magnetic Flux and Voltage



Four Circuit parameters

Charge and current, Magnetic Flux and Voltage



Electromagnetic Oscillator

