

PHYS 514

Review Lecture – Circuit QED

Week - 10

Node flux and Lagrange

For linear oscillator

$$\mathcal{L} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_0[\dot{\Phi}_1 - \dot{\Phi}_2]^2 - \frac{1}{2L_1}\Phi_1^2 - \frac{1}{2L_2}\Phi_2^2$$

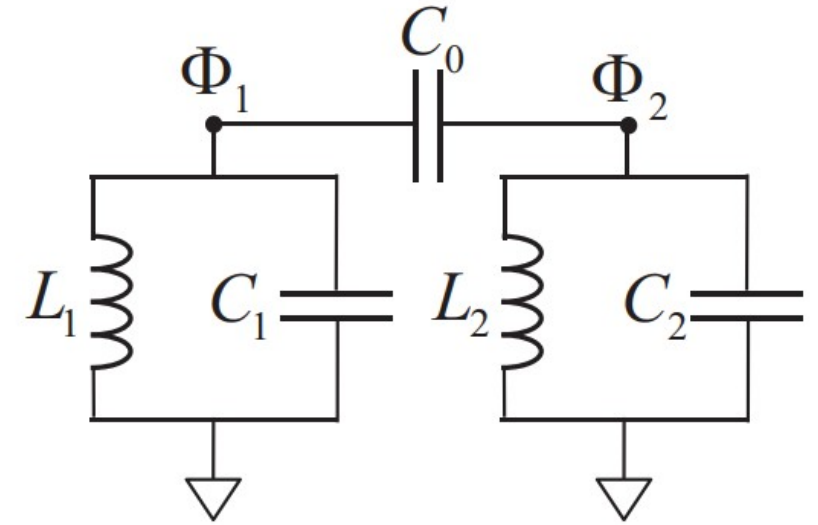
$$H = \sum_i Q_i \dot{\Phi}_i - \mathcal{L} \quad \text{where} \quad Q_i = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i}$$

New

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}^T C \dot{\Phi} - \frac{1}{2}\Phi^T L^* \Phi$$

Where C is capacitance matrix

L^* is susceptance (matrix) – inverse of inductance $L^* = \frac{1}{L}$



For given circuit

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$



$$C = \begin{pmatrix} C_1 + C_0 & -C_0 \\ -C_0 & C_2 + C_0 \end{pmatrix}$$

Diagonal - Sum of capacitance connected to node

Off diagonal - negative sum of capacitances between two nodes

$$L^* = \begin{pmatrix} \frac{1}{L_{11}} & \frac{1}{L_{12}} & \frac{1}{L_{13}} \\ \frac{1}{L_{21}} & \frac{1}{L_{22}} & \frac{1}{L_{23}} \\ \frac{1}{L_{31}} & \frac{1}{L_{32}} & \frac{1}{L_{33}} \end{pmatrix}$$



$$L^* = \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{pmatrix}$$

Diagonal - Sum of susceptance connected to node

Off diagonal - negative sum of susceptances between two nodes

Node flux and Lagrange and Hamiltonian For linear oscillator

linear oscillator – containing linear inductor and capacitor

$$P.E = U = \frac{1}{2} \sum_{i,j} \frac{(\Phi_i - \Phi_j)^2}{l_{ij}}$$

$$K.E = T = \frac{1}{2} \sum_{i,j} c_{ij} (\dot{\Phi}_i - \dot{\Phi}_j)^2$$

$$\mathcal{L} = T(\dot{\Phi}) - U(\Phi)$$

Lagrange

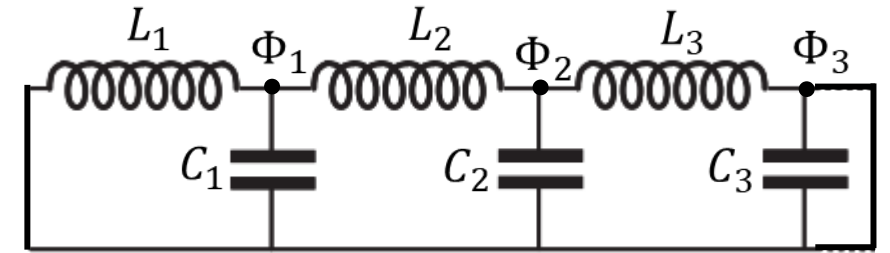
$$\mathbf{Q} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \mathbf{C} \dot{\Phi}$$

Conjugate momenta

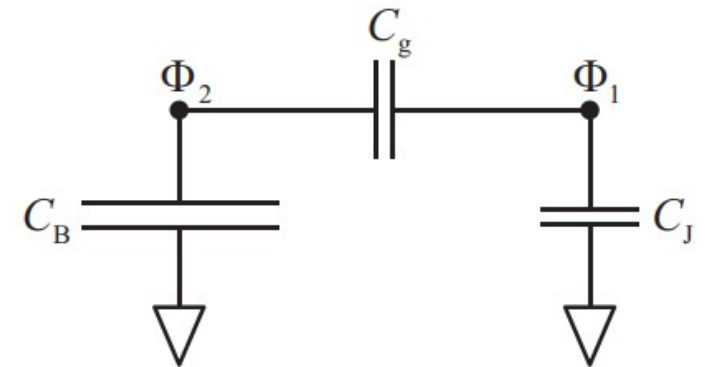
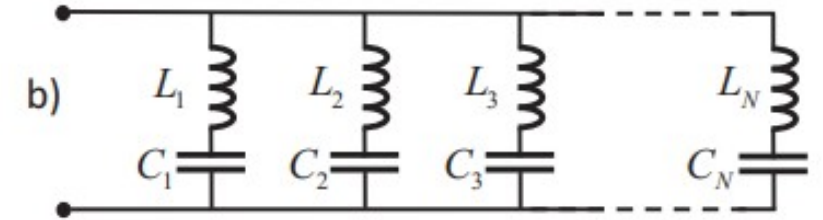
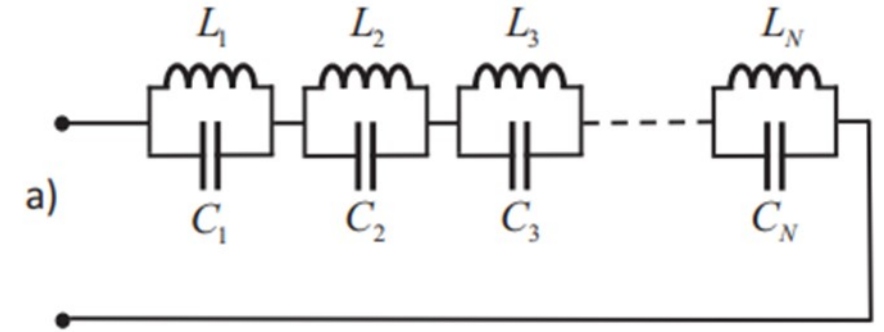
$$H = \frac{1}{2} \Phi^T L^* \Phi + \frac{1}{2} \mathbf{Q}^T \mathbf{C}^{-1} \mathbf{Q}$$

Hamiltonian

Node flux and Lagrange and Hamiltonian For linear oscillator



Node flux and Lagrange and Hamiltonian For linear oscillator



Node flux and Lagrange and Hamiltonian For non-linear oscillator

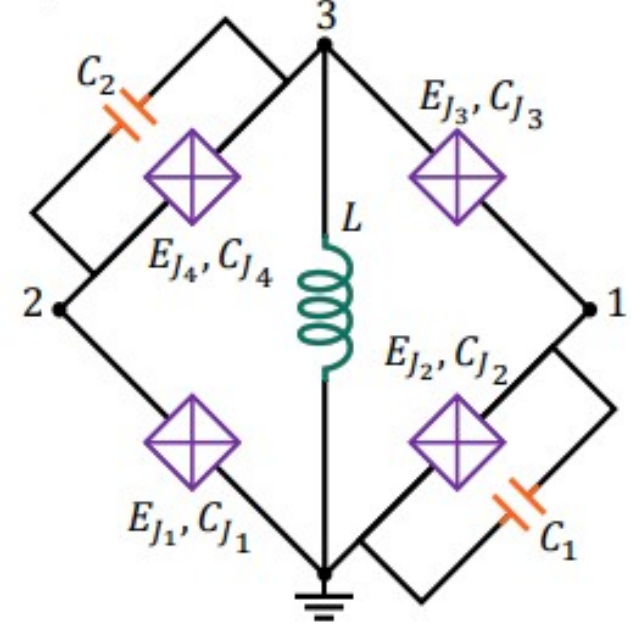
Define flux at node 1, 2, and 3 - Φ_1 , Φ_2 , and Φ_3

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(C_1 + C_{J2})\dot{\Phi}_1^2 + \frac{1}{2}(C_{J1})\dot{\Phi}_2^2 + \frac{1}{2}(C_2 + C_{J4})(\dot{\Phi}_3 - \dot{\Phi}_3)^2 + \frac{1}{2}C_{J3}(\dot{\Phi}_3 - \dot{\Phi}_3)^2 - \frac{1}{2L}\Phi_3^2 \\ &\quad - \frac{1}{2L_{J1}}\Phi_2^2 - \frac{1}{2L_{J2}}\Phi_1^2 - \frac{1}{2L_{J4}}(\Phi_3 - \Phi_2)^2 - \frac{1}{2L_{J3}}(\Phi_3 - \Phi_1)^2 \end{aligned}$$

Here $L_{J's}$ are NOT constant, we will deal with such situations with more concepts of circuit theory

$$C = \begin{pmatrix} C_1 + C_{J2} + C_{J3} & 0 & -C_{J3} \\ 0 & C_2 + C_{J1} + C_{J4} & -C_2 - C_{J4} \\ -C_{J3} & -C_2 - C_{J4} & C_2 + C_{J3} + C_{J4} \end{pmatrix}$$

$$L^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{l} \end{pmatrix}$$



Hamiltonian of Oscillator

$$H = Q\dot{\phi} - \mathcal{L} = \frac{1}{2C}Q^2 + \frac{\phi^2}{2L}.$$

$$[\hat{Q}, \hat{\phi}] = -i\hbar$$

$$\begin{aligned}\hat{a} &= +i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi} \\ \hat{a}^\dagger &= -i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi}\end{aligned}$$

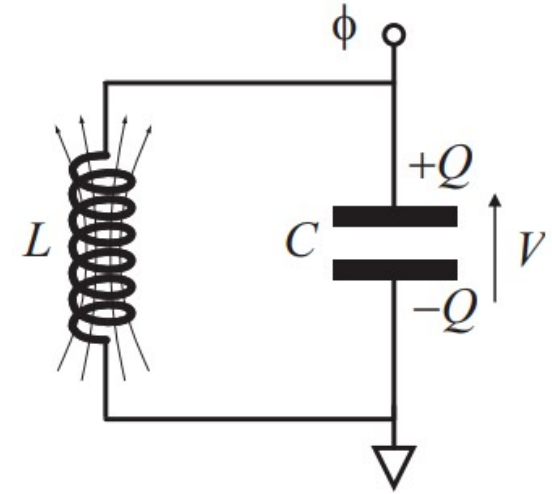


$$\begin{aligned}\hat{Q} &= -iQ_{\text{ZPF}}(\hat{a} - \hat{a}^\dagger) \\ \hat{\phi} &= \Phi_{\text{ZPF}}(\hat{a} + \hat{a}^\dagger),\end{aligned}$$

$$Q_{\text{ZPF}} = \sqrt{\frac{C\hbar\Omega}{2}} = \sqrt{\frac{\hbar}{2Z}}$$

$$\Phi_{\text{ZPF}} = \sqrt{\frac{L\hbar\Omega}{2}} = \sqrt{\frac{\hbar Z}{2}},$$

Where $Z = \sqrt{\frac{L}{C}}.$



$$[\hat{a}, \hat{a}^\dagger] = 1.$$

$$H = \frac{\hbar\Omega}{2} \{\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger\} = \hbar\Omega \left\{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \right\},$$

$$\begin{aligned}\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{n}|n\rangle &= \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle\end{aligned}$$

$$\langle n|\hat{Q}|n\rangle = 0$$

$$\langle n|\hat{\phi}|n\rangle = 0$$

and

$$\langle 0|\hat{Q}^2|0\rangle = Q_{\text{ZPF}}^2$$

$$\langle 0|\hat{\phi}^2|0\rangle = \Phi_{\text{ZPF}}^2.$$

$$R_Q \equiv \frac{h}{(2e)^2} \approx 6,453.20 \text{ Ohms}, \quad (2.48)$$

we can define a dimensionless characteristic impedance

$$z \equiv Z/R_Q, \quad (2.49)$$

to obtain

$$Q_{\text{ZPF}} = (2e) \sqrt{\frac{1}{4\pi z}} \quad (2.50a)$$

$$\Phi_{\text{ZPF}} = \Phi_0 \sqrt{\frac{z}{4\pi}}, \quad (2.50b)$$

where

$$\Phi_0 \equiv \frac{h}{2e} \quad (2.51)$$

is the superconducting flux quantum. Notice that the usual uncertainty product is obeyed.

$$Q_{\text{ZPF}} \Phi_{\text{ZPF}} = \frac{\hbar}{2}. \quad (2.52)$$

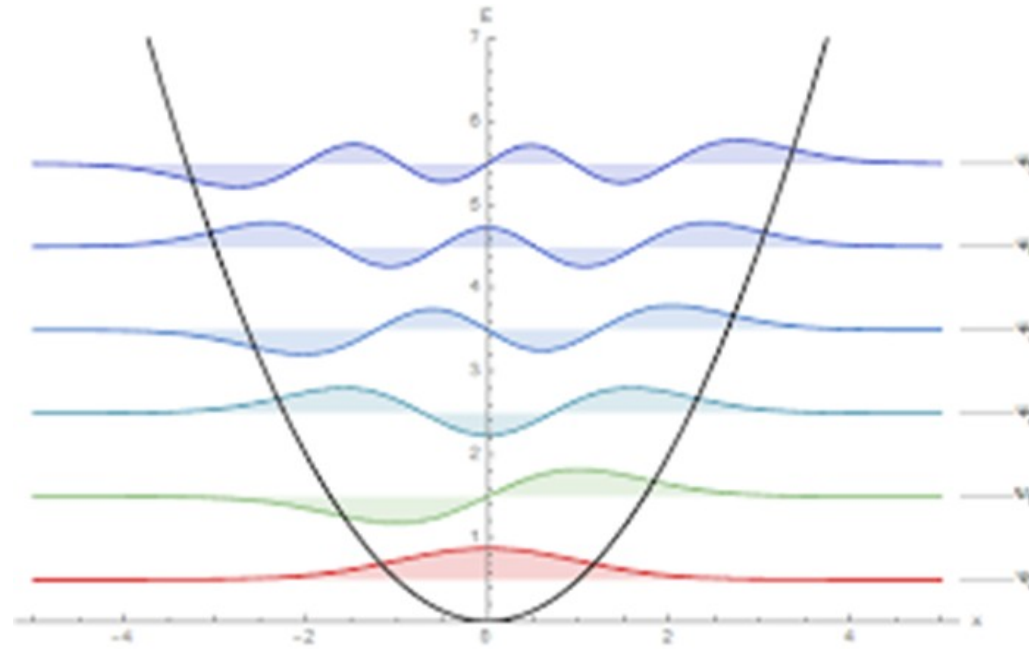
The voltage is an important physical variable and the voltage operator is given by

$$\begin{aligned} \hat{V} &= \frac{d\hat{\phi}}{dt} = \frac{i}{\hbar} [H, \hat{\phi}] \\ &= \frac{1}{C} \hat{Q} = -i \sqrt{\frac{\hbar \Omega}{2C}} (\hat{a} - \hat{a}^\dagger) = -i V_{\text{ZPF}} (\hat{a} - \hat{a}^\dagger), \end{aligned} \quad (2.53)$$

where

$$V_{\text{ZPF}} = \Omega \Phi_{\text{ZPF}} = \Omega \Phi_0 \sqrt{\frac{z}{4\pi}}. \quad (2.54)$$

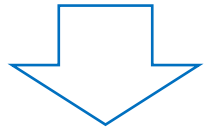
Wave Function $|\psi\rangle$ for Harmonic Oscillator



Hamiltonian of E &M wave /Oscillator

$$H = \frac{1}{2} [p^2(t) + \omega_c^2 q^2(t)]$$

$q(t)$ is position and $p(t)$ is (conjugate) momentum



Classical to Quantum

$$\hat{H} = \frac{1}{2} [\hat{p}^2(t) + \omega_c^2 \hat{q}^2(t)]$$

Every physical quantity becomes an operator

Operators can be expressed as matrix and their expectation values can be determined for given state

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2\omega_c}}(\omega_c \hat{q} + i\hat{p}), \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\omega_c}}(\omega_c \hat{q} - i\hat{p}),\end{aligned}$$



$$\hat{H} = \omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) = \omega_c \left(\hat{n} + \frac{1}{2} \right)$$

$$\hat{H}|n\rangle = E_n|n\rangle, \quad n = 0, 1, 2, \dots$$

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}$$

Time evolution of ladder operator

Coherent State

$$|\psi\rangle = |\alpha\rangle = \sum_n c_n |n\rangle, \quad c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}},$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Eigen state of lowering operator $\hat{a}$

Josephson Junction

$$I = I_c \sin(\Delta\varphi)$$

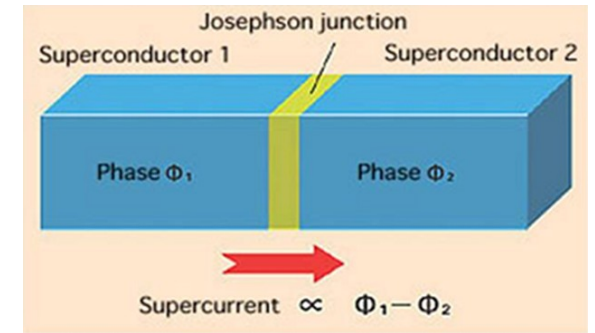
$$V = \frac{\hbar}{2e} \frac{d\Delta\varphi}{dt}$$

Josephson Equations

$$L = \frac{\hbar}{2e I_c \cos \Delta\varphi} \equiv L_J$$

Non-linear Josephson Inductance

$$E = -E_J(1 - \cos(\Delta\varphi)) \quad \text{Josephson energy}$$



Hamiltonian of JJ

H for Mechanical Osc.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

H for LC Osc.

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega_{01}^2\hat{\phi}^2$$

K.E
P.E

H for JJ Osc.

$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_J \left(1 - \cos\left(\frac{2\pi\hat{\Phi}}{\Phi_0}\right)\right)$$

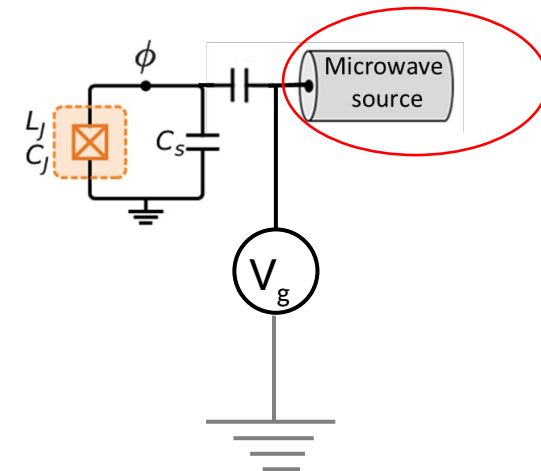
$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_J (1 - \cos(\Delta\hat{\Phi}))$$

Replacing $\hat{Q}$ by “charge number operator” $\hat{n}$

$$\hat{H} = \frac{(2e)^2}{2C}\hat{n}^2 - E_J(1 - \cos(\Delta\hat{\Phi}))$$

$$\hat{H} = 4E_c\hat{n}^2 - E_J\cos(\Delta\hat{\Phi}) \approx 4E_c\hat{n}^2 - E_J(1 - \cos(\Delta\hat{\Phi}))$$

$$\hat{H} = 4E_c(\hat{n} - n_g)^2 - E_J\cos(\Delta\hat{\Phi}) \quad \text{Including charge noise}$$



$$\hat{H} = 4E_c \hat{n}^2 - E_J \cos(\Delta\hat{\phi})$$

$$\hat{H} = 4E_c (\hat{n} - n_g)^2 - E_J \cos(\Delta\hat{\phi})$$

Expanding cosine function and considering first three terms in the expansion.

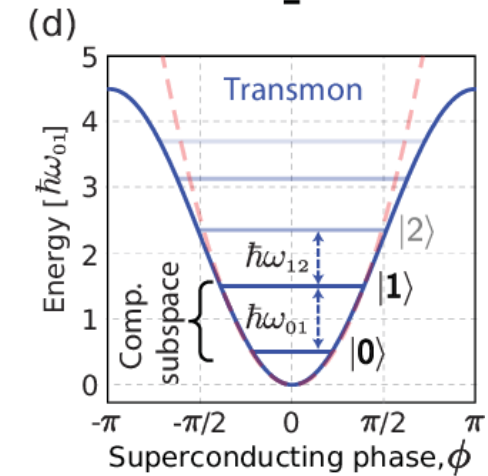
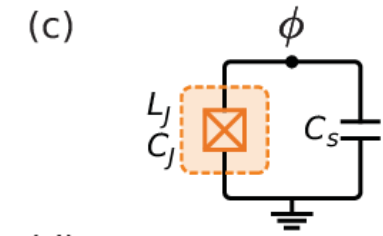
As usual after introducing suitable ladder operator

$$\begin{aligned}\hat{H}_{\text{trans}} &= \omega_J \hat{b}^\dagger \hat{b} - \frac{E_C}{12} (\hat{b} + \hat{b}^\dagger)^4 + \dots \\ &= \omega_J \hat{b}^\dagger \hat{b} - \frac{E_C}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + 2\hat{b}^\dagger \hat{b}) + \dots,\end{aligned}$$

$$\begin{aligned}\hat{H}_{\text{trans}} &= (\omega_J - E_C) \hat{b}^\dagger \hat{b} - \frac{E_C}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \\ &= \omega_{01} \hat{b}^\dagger \hat{b} + \frac{\alpha}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b},\end{aligned}$$

Where ω_{01} is qubit transition frequency

and α is define the non-linearity of the osc.



$$\begin{aligned}4E_c &\leftrightarrow \frac{1}{2C} \\ \frac{E_J}{2} &\leftrightarrow \frac{1}{2L} \\ \omega_J = \sqrt{8E_J E_c} &\leftrightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}\end{aligned}$$

For more details of expressions, see next slide (14)

Transmon Regime Harmonic Oscillator

$$\hat{H} = 4E_c (\hat{n})^2 + \frac{1}{2} E_J \hat{\phi}^2 - \frac{1}{24} E_J \hat{\phi}^4$$

$$\Rightarrow \hat{H} = \frac{\hat{n}^2}{2C'} + \frac{\hat{\phi}^2}{2L'} - \frac{1}{24} E_J \hat{\phi}^4 \dots (1)$$

First two terms Hamiltonian is same as of LC osc. (eqs. On the right)

Therefore, for first two term

$$\hat{n} = i\hbar \left(\frac{E_J}{32E_c} \right)^{1/4} (\hat{b}^\dagger - \hat{b}) \quad \hat{b}^\dagger, \hat{b} \text{ are ladder operators for Transmon}$$

$$\hat{\phi} = \hbar \left(\frac{2E_c}{E_J} \right)^{1/4} (\hat{b}^\dagger + \hat{b})$$

Substituting above back in eq. (1)

$$\hat{H} = \sqrt{8E_c E_J} \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + 2\hat{b}^\dagger \hat{b})$$

$$\hat{H} = (\sqrt{8E_c E_J} - E_c) \hat{b}^\dagger \hat{b} - \frac{E_c}{2} (\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} +)$$

Drop the term where the number of $\hat{b}^\dagger$ and $\hat{b}$ term are not same (rotating wave approx.).

$$\Omega_J = \frac{1}{\hbar} \sqrt{8E_c E_J} = \frac{1}{\sqrt{L_J C}}$$

Josephson Plasma frequency

From Ch. 1

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \hat{\phi}^2$$

$$\hat{Q} = i \sqrt{\frac{\hbar}{2Z_0}} (\hat{a}^\dagger - \hat{a})$$

$$\hat{\phi} = i \sqrt{\frac{\hbar Z_0}{2}} (\hat{a}^\dagger + \hat{a})$$

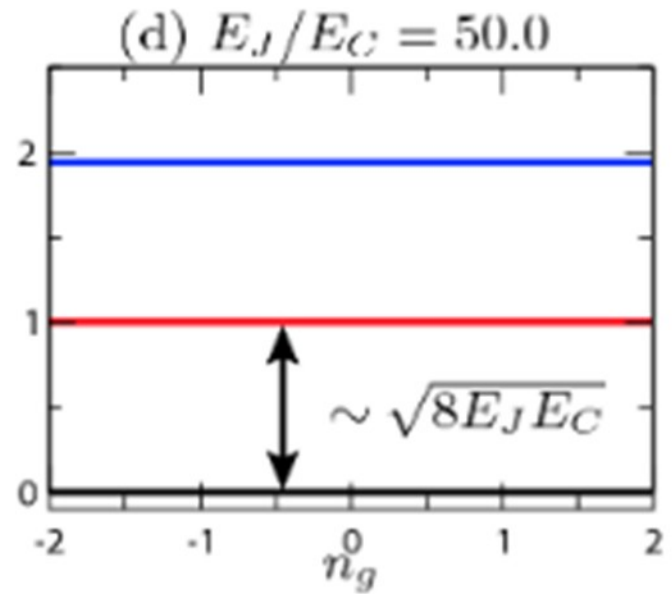
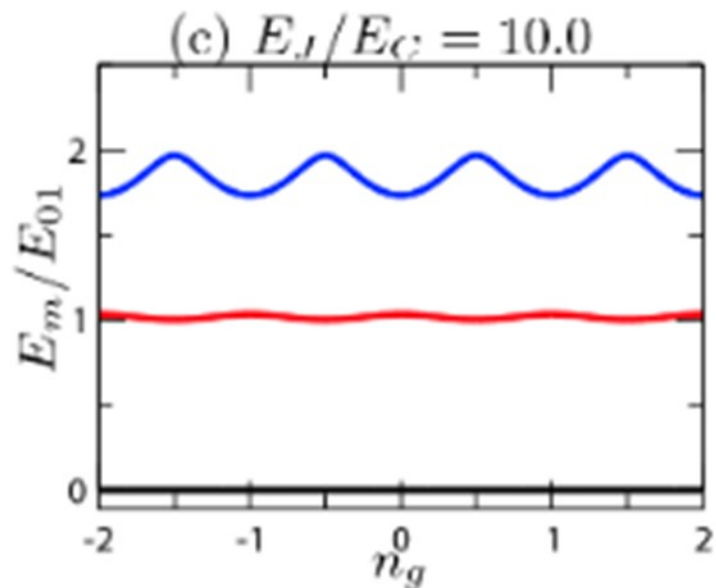
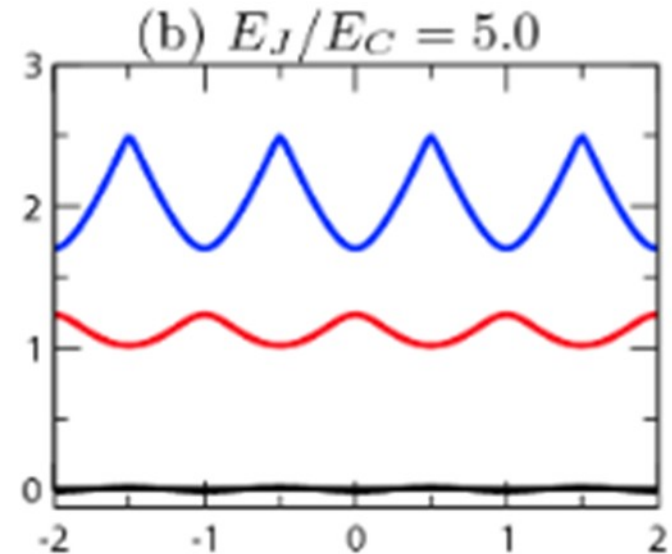
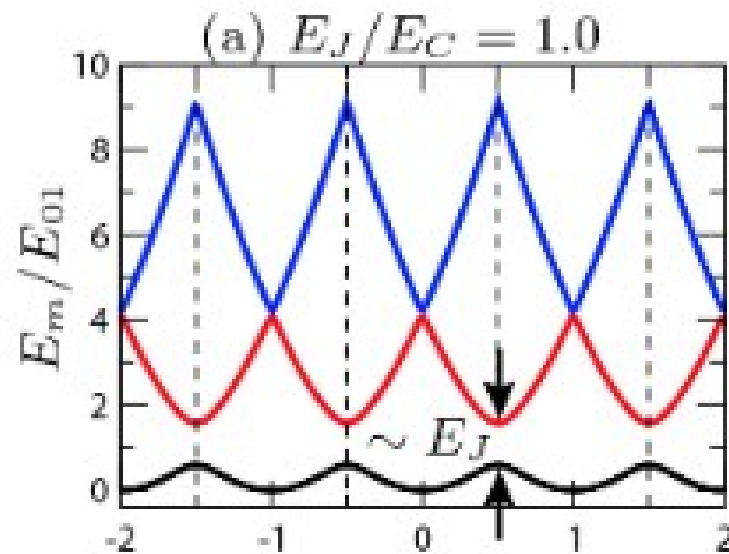
$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\hat{H} = \hbar \omega (\hat{a}^\dagger \hat{a} + 1)$$

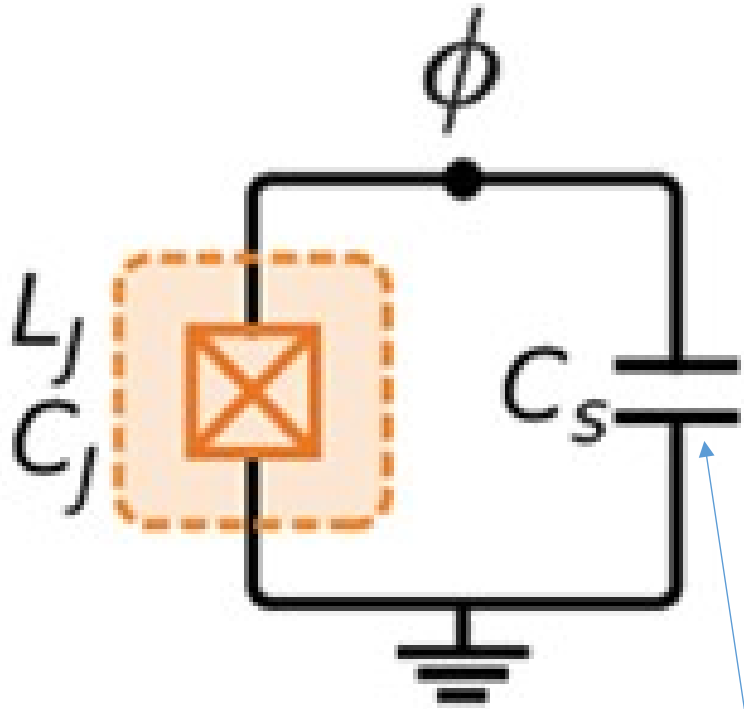
Transmon Qubit

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos(\Delta\hat{\phi})$$

Energy level of JJ-based Harmonic Osc are extremely sensitive to charge noise (from control unit and/or environment etc.)



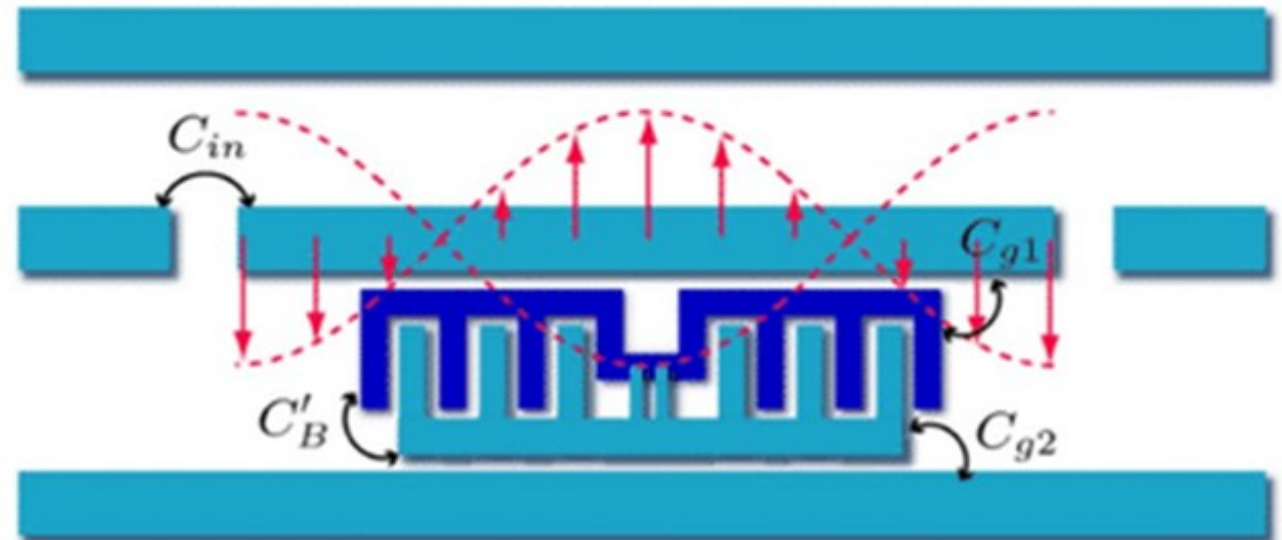
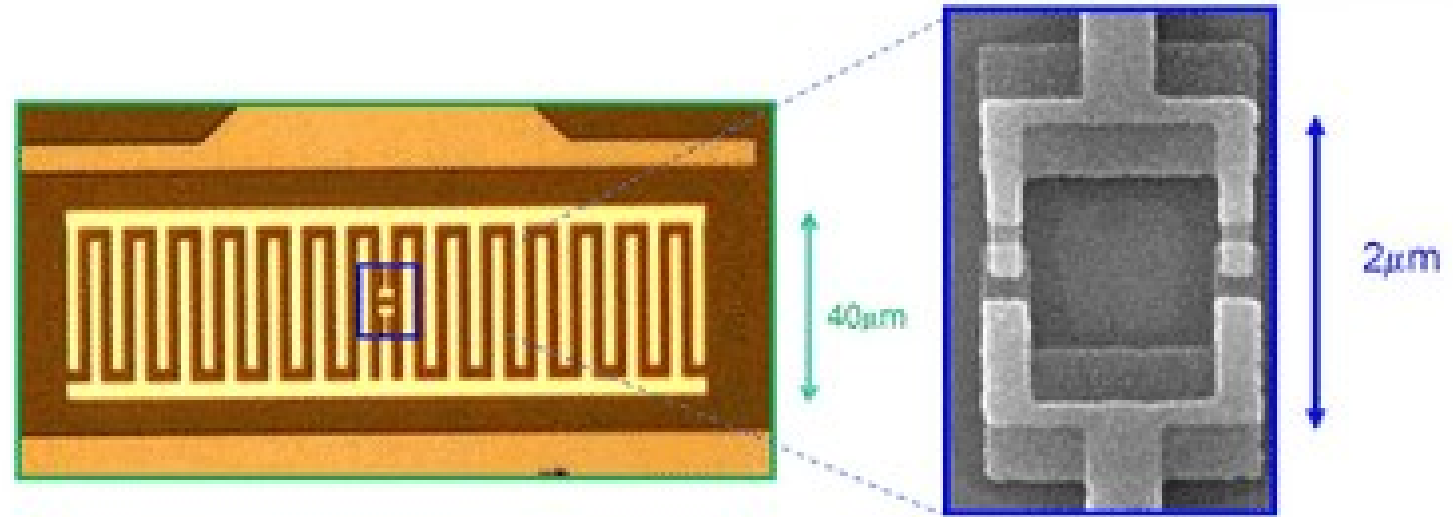
Transmon Qubit



Capacitors of
very large value

$$E_c = \frac{q^2}{2C}$$

to keep the energy stored by the
capacitor much less than is E_J



Qubit-Cavity Interaction

$$H_{sys} = H_q + H_c + H_I$$

H_q, H_c are known, including eigen states and energy eigen values

Now are interested in energy eigen values and eigen states of H_{sys}

$$H_{int} = -g(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-)$$

$$H_{\text{Rabi}} = \omega_c(\hat{a}^\dagger \hat{a} + \frac{1}{2}) - \frac{1}{2}\omega_q \sigma_z - g(\hat{a} + \hat{a}^\dagger)(\sigma_- + \sigma_+).$$

Bare states no longer are the energy eigenstates for the system when the qubit and cavity interact ($g \neq 0$).

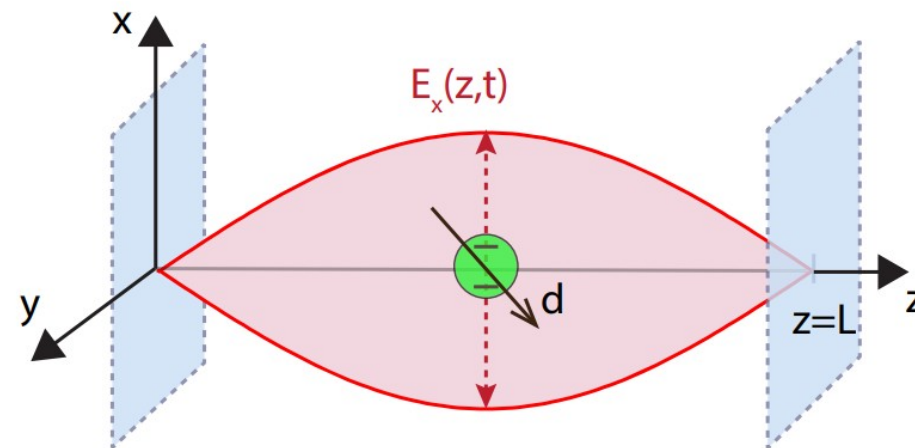
In the case of no interaction between qubit and cavity ($g = 0$) **the eigenstates of the qubit-cavity system are simply the tensor product of the cavity and qubit eigenstates** $\{|g\rangle|n\rangle, |e\rangle|n\rangle\}$ which are called **bare states or the bare basis** with eigen energies are simple sum i.e. $\{\pm\omega_q/2 + \omega_c(n + 1/2)\}$.

Other possible bare states

$|g\rangle|0\rangle \rightarrow$ qubit in ground state, no photons in the cavity

$|g\rangle|n+1\rangle \rightarrow$ qubit in ground state, $n+1$ photons in the cavity

$|e\rangle|n\rangle \rightarrow$ qubit in excited state, n photons in the cavity



Qubit-Cavity Interaction

Now bare states no longer are the energy eigenstates for the system when the qubit and cavity interact ($g \neq 0$).

Yet, we can represent the **total Hamiltonian in the bare basis and attempt to diagonalize it to find its eigenstates and eigenvalues**. Before we do this, we simplify the interaction Hamiltonian by the rotating wave approximation (RWA). **This approximation is valid in most practical situations** where the **coupling strength** is much less than both the qubit and cavity frequency, $g \ll \omega_c, \omega_q$ and also $|\omega_c - \omega_q| \ll |\omega_c + \omega_q|$

Under RWA interaction term reduces to

$$H_{int} \Rightarrow \hat{a}^\dagger \sigma_- + \hat{a} \sigma_+ + \cancel{\hat{a}^\dagger \sigma_+} + \cancel{\hat{a} \sigma_-}$$

Excitation and decay of both the qubit and the cavity are highly improbable.

$$H_{\text{Rabi}} = \omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \frac{1}{2} \omega_q \sigma_z - g(\hat{a} + \hat{a}^\dagger)(\sigma_- + \sigma_+).$$

$$H_{\text{JC}} = \omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \frac{1}{2} \omega_q \sigma_z - g(\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+)$$

Jaynes-Cummings Hamiltonian

Eigen states of above Hamiltonian

$$|0, -\rangle = |g\rangle|0\rangle$$

$$|n, -\rangle = \cos(\theta_n)|g\rangle|n+1\rangle - \sin(\theta_n)|e\rangle|n\rangle$$

$$|n, +\rangle = \sin(\theta_n)|g\rangle|n+1\rangle + \cos(\theta_n)|e\rangle|n\rangle$$

For $\theta_n = 0$ or 90° the states are bare states

For $\theta_n \neq 0$ or 90° , states are superpositions states

Qubit-cavity Eigen states and Energy Eigen Values

Energy eigen values

$$E_{\mp} = (n+1)\omega_c \mp \frac{1}{2}\sqrt{4g^2(n+1) + \Delta^2}.$$

Where $\Delta = \omega_q - \omega_c$

When $\Delta \rightarrow \infty \Rightarrow \theta_n = 0$

$$|n, -\rangle = |g\rangle|n+1\rangle$$

$$|n, +\rangle = |e\rangle|n\rangle$$

When $\Delta \rightarrow 0 \Rightarrow \theta_n = \pi/4$

$$|n, \mp\rangle = \frac{1}{\sqrt{2}} (|g\rangle|n+1\rangle \mp |e\rangle|n\rangle)$$

Eigen States

$$|0, -\rangle = |g\rangle|0\rangle$$

$$|n, -\rangle = \cos(\theta_n)|g\rangle|n+1\rangle - \sin(\theta_n)|e\rangle|n\rangle$$

$$|n, +\rangle = \sin(\theta_n)|g\rangle|n+1\rangle + \cos(\theta_n)|e\rangle|n\rangle$$

where $\theta_n = \frac{1}{2} \tan^{-1}(2g\sqrt{n+1}/\Delta)$

$$\begin{aligned}
|n, -\rangle &= \cos(\theta_n)|g\rangle|n+1\rangle - \sin(\theta_n)|e\rangle|n\rangle \\
|n, +\rangle &= \sin(\theta_n)|g\rangle|n+1\rangle + \cos(\theta_n)|e\rangle|n\rangle
\end{aligned}$$

$$\theta_n = \frac{1}{2} \tan^{-1}(2g\sqrt{n+1}/\Delta)$$

Where $\Delta = \omega_q - \omega_c$

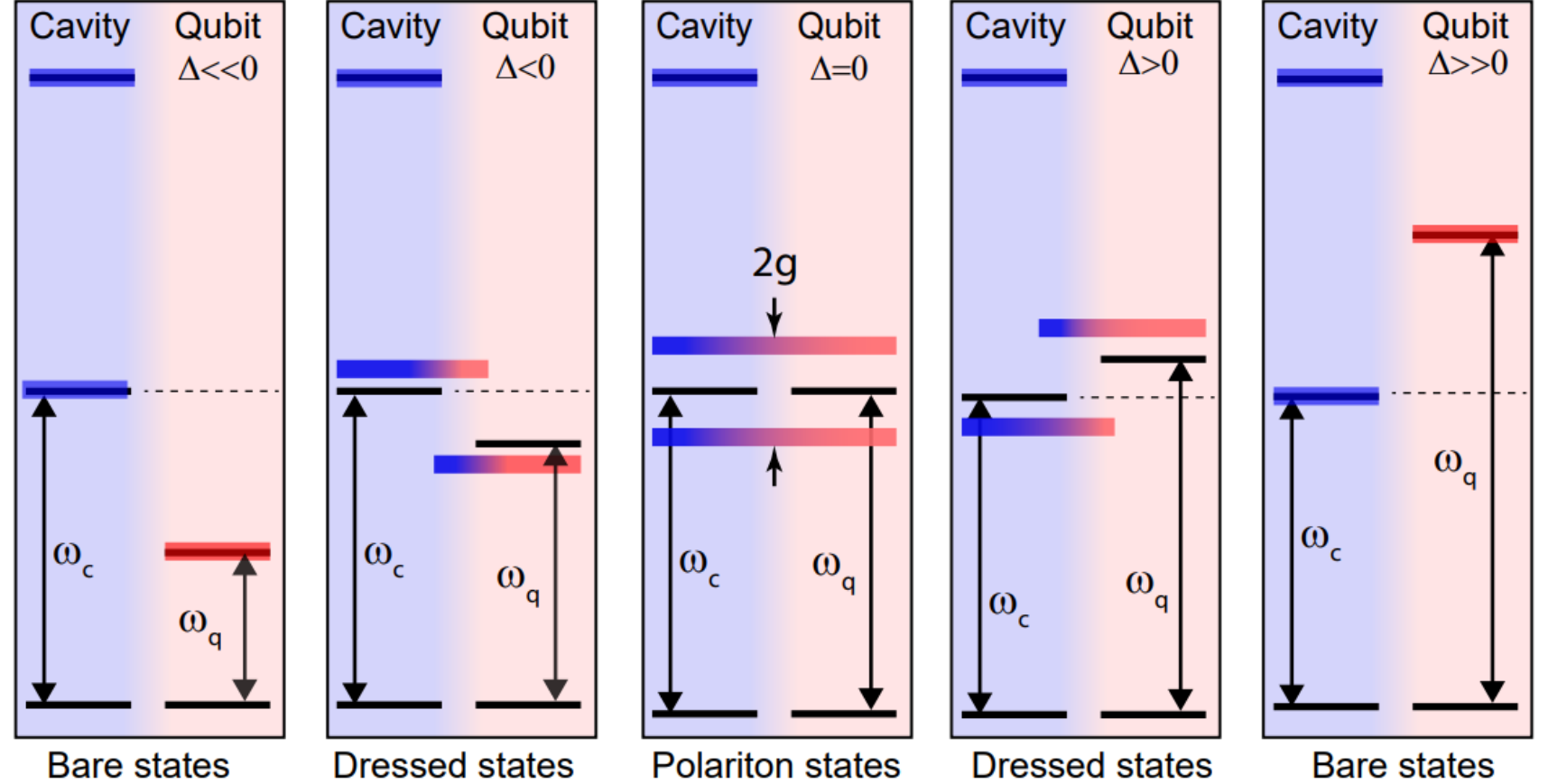


Figure 2.11: **Dressed states vs bare states:** The panels illustrate the dressed states of the qubit-cavity system for different qubit-cavity detunings in comparison with the bare states (refer to the main text for a more detailed description). Note that this illustration is not accurate and lacks some details but we rather to avoid them here.

It is convenient to plot transition energy versus detuning since (during experiment) we normally characterize the system by measuring the transition frequencies by doing spectroscopy. For example, when $n = 0$ we have

$$E_{\mp} - E_g = \omega_c \mp \frac{1}{2} \sqrt{4g^2 + \Delta^2} + \Delta/2. \quad \text{Where } E_g = \frac{\Delta}{2}$$

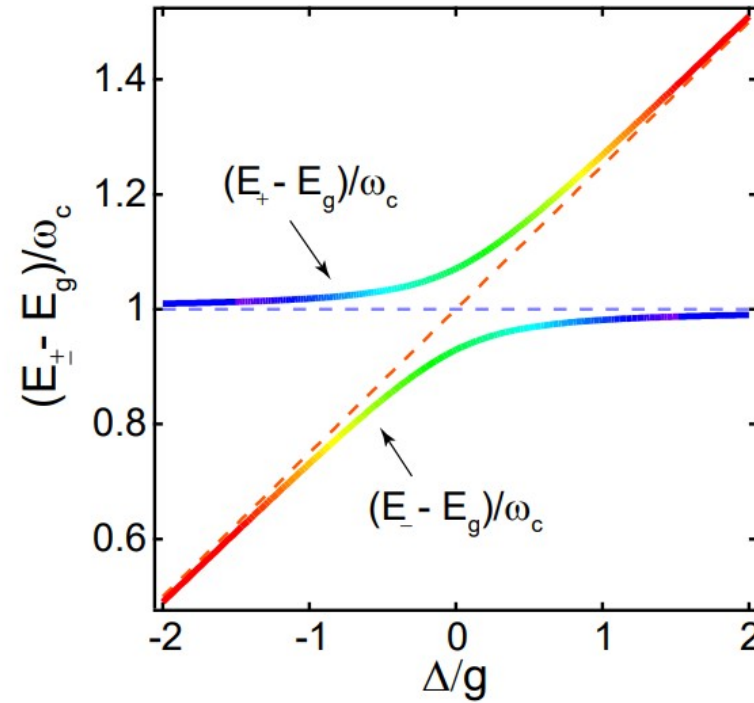


Figure 2.12: **Avoided crossing:** The transition energy from higher and lower dressed states to the ground state versus the detuning Δ . The transition energy is scaled by the energy of the cavity ω_c and the detuning is scaled by the coupling rate g . The dashed lines indicate the bare states' transition. Note that you can somehow see a similar avoided-crossing curve in Figure 2.11 by connecting the upper (lower) dressed states in different detunings together.

Qubit-Cavity System under Dispersive Approximation

$$\Delta \gg g; \Delta = \omega_q - \omega_c$$

In such situations, **the interaction is relatively weak**. In principle, **in this regime, the cavity and qubit do not directly exchange energy** unlike what we explicitly have in the interaction term in the JC Hamiltonian

$$H_{\text{JC}} = \omega_c(\hat{a}^\dagger \hat{a} + \frac{1}{2}) - \frac{1}{2}\omega_q \sigma_z - g(\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+).$$

Under dispersive approximation (after doing some mathematics!), the Hamiltonian reduces to

$$\hat{H}_{\text{dis}} = \omega_c \hat{a}^\dagger \hat{a} - \frac{1}{2}\omega_q \sigma_z - \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \sigma_z.$$

Substituting $\chi = \frac{g^2}{\Delta}$ and after rearrangement

$$\hat{H}_{\text{dis}} = (\omega_c - \chi \sigma_z) \hat{a}^\dagger \hat{a} - \frac{1}{2}\omega_q \sigma_z,$$

We now σ_z can be +1 or -1 depending upon the state of qubit

Therefore depending upon the state of qubit there is a 'shift' in cavity resonant frequency

Cavity resonant frequency measurement can read the state of qubit

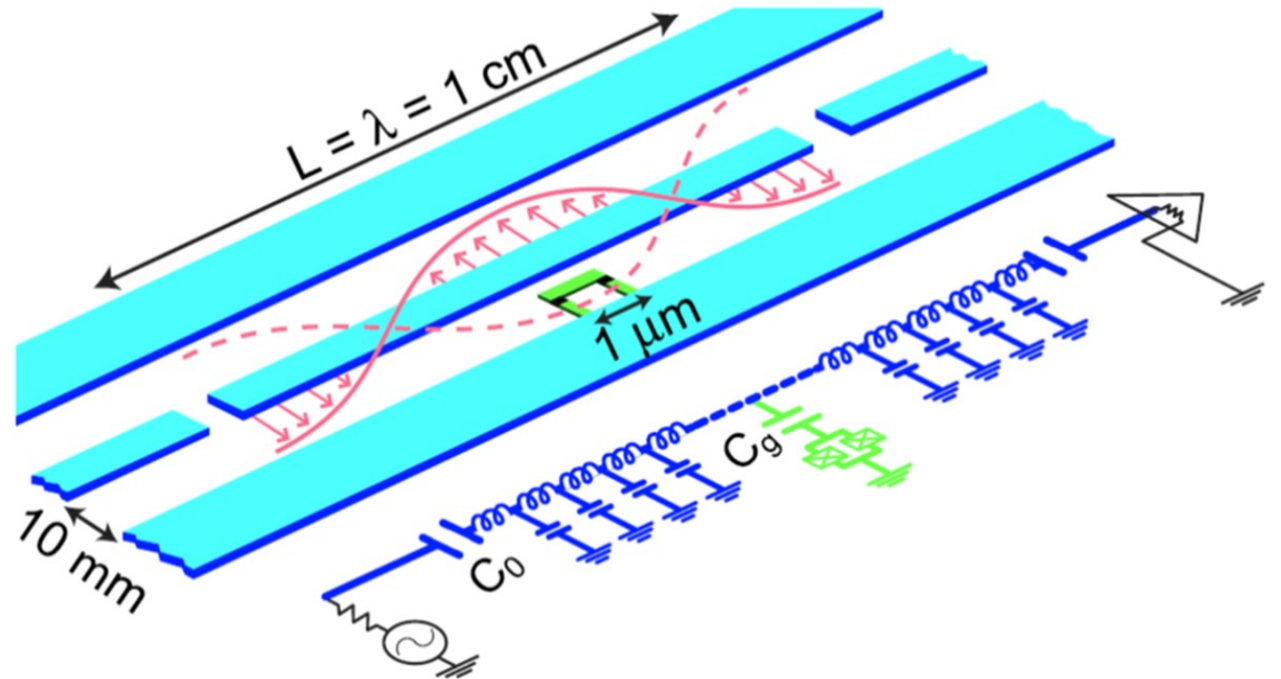
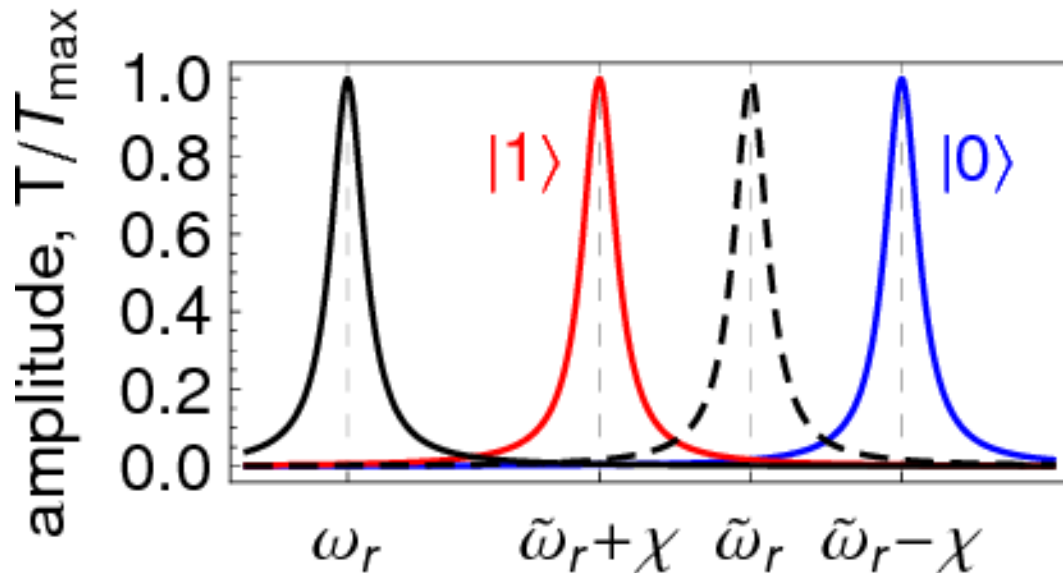
How to Read the State of a Qubit

Coupling between Resonator (LC circuit) and qubit (L,C -circuit)

Resonator has its own resonant frequency (can be controlled by LC i.e. by material and its dimension)

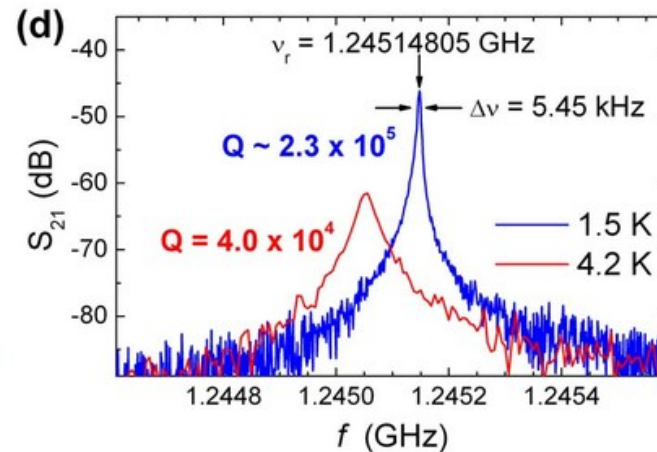
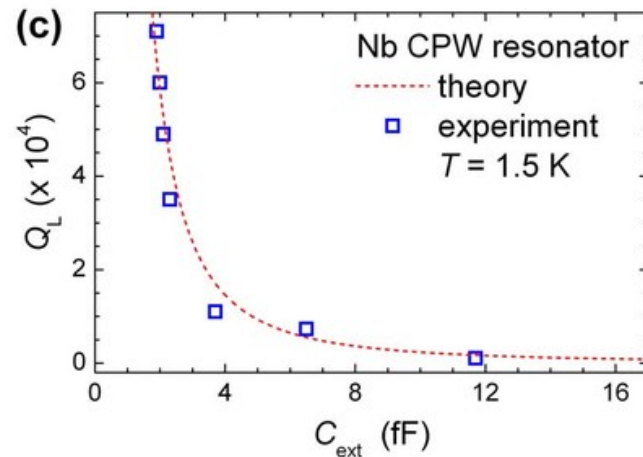
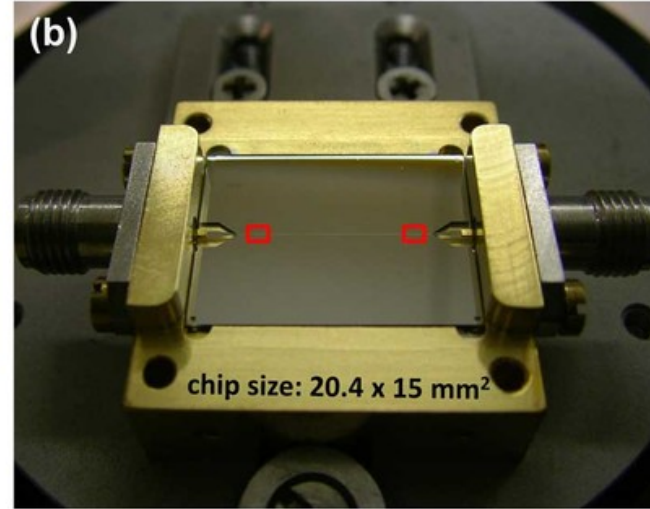
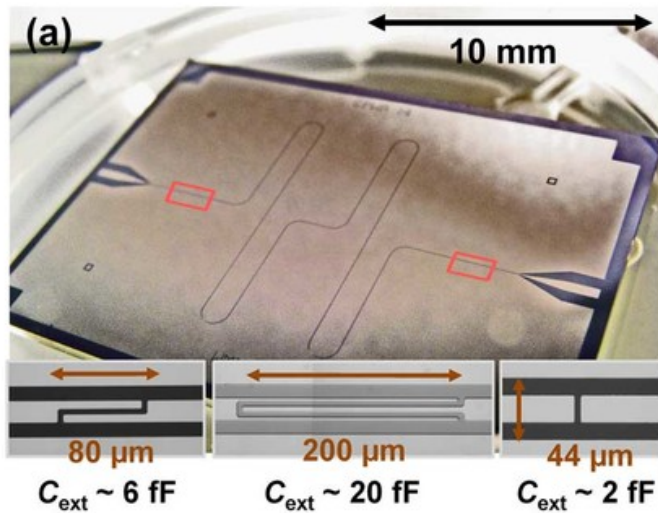
Qubit also has its own resonant frequency (can be controlled by JJ properties and Shunt capacitor)

When two resonators couple, there will be a shift in resonant frequency

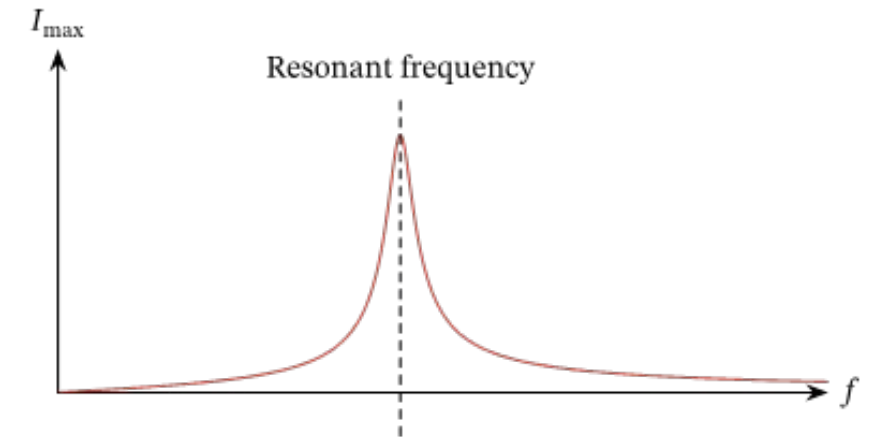


Basic idea of VNA (vector network analyzer) measurements

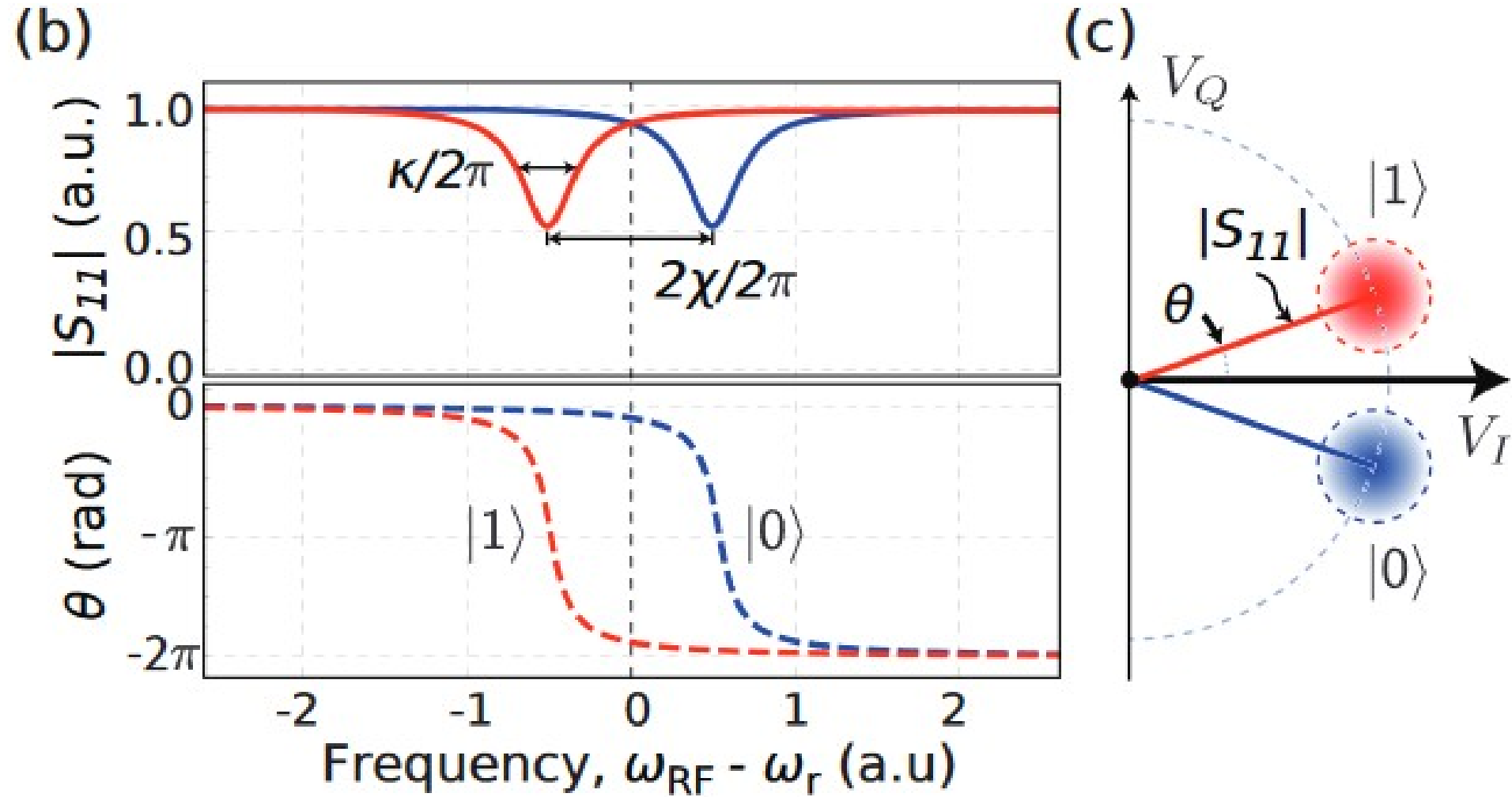
S_{11} (reflected) or S_{21} (transmitted) scattering parameter measurements



At resonance transmitted (reflected) signal will be maximum (minimum)



Shift in resonant frequency of cavity/Resonator – measuring the state of qubit



NDM – non demolition measurement

Rabi Oscillations

$$|\psi\rangle = C_g|g\rangle + C_e|e\rangle$$

Qubit state at $t = 0$

$$|\psi(t)\rangle = C_g(t)e^{+i\frac{\omega_q}{2}t}|g\rangle + C_e(t)e^{-i\frac{\omega_q}{2}t}|e\rangle,$$

Qubit state at time 't'

Using time dependent SWE

$$C_g(t) = \frac{e^{-i\frac{\Delta_d}{2}t}}{\Omega_R} \left(\Omega_R \cos\left(\frac{\Omega_R}{2}t\right) + i\Delta_d \sin\left(\frac{\Omega_R}{2}t\right) \right)$$
$$C_e(t) = i\frac{A e^{+i\frac{\Delta_d}{2}t}}{\Omega_R} \sin\left(\frac{\Omega_R}{2}t\right),$$

These equations can be used to derive an analytical expression for the time-dependent probabilities of the qubit's ground and excited states.

where

$$\Delta_d = \omega_q - \omega_d \text{ and } \Omega_R = \sqrt{A^2 + \Delta_d^2}.$$

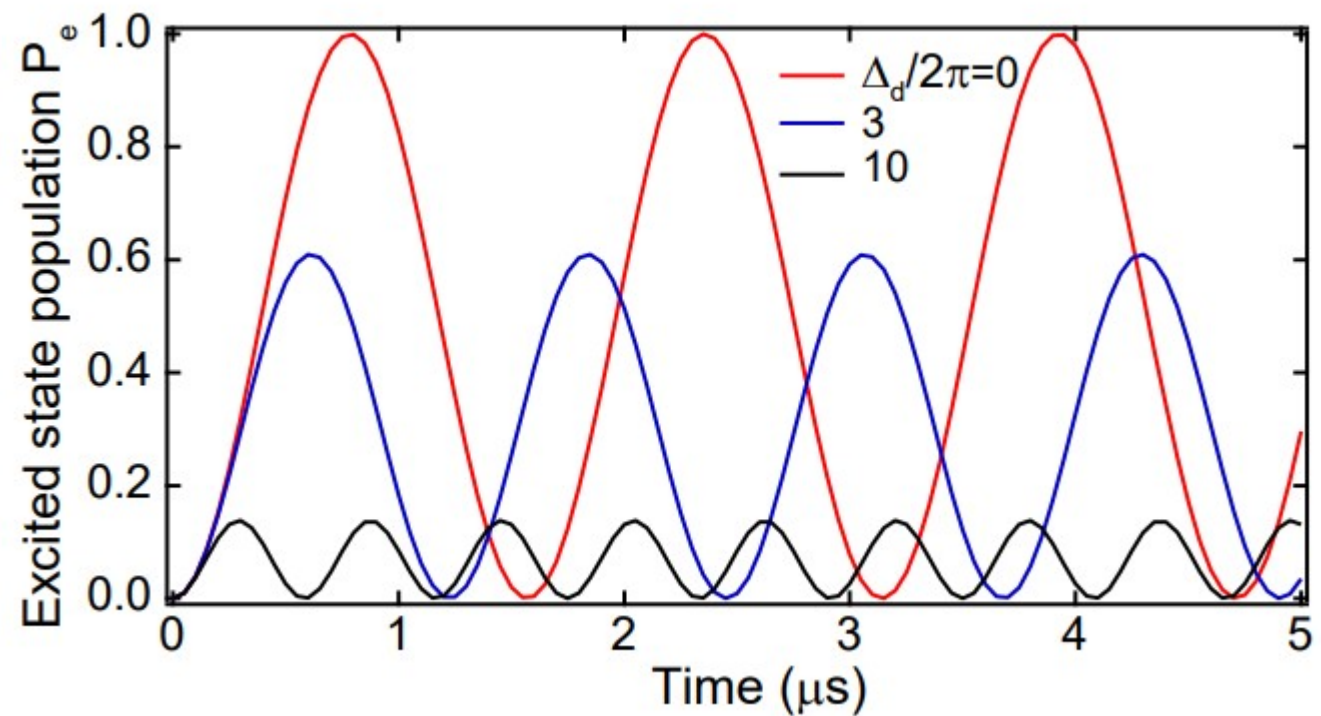
$A = Ed$; E is electric field of the cavity/resonator and d is dipole moment of the qubit

$$C_g(t) = \frac{e^{-i\frac{\Delta_d}{2}t}}{\Omega_R} \left(\Omega_R \cos\left(\frac{\Omega_R}{2}t\right) + i\Delta_d \sin\left(\frac{\Omega_R}{2}t\right) \right)$$

$$C_e(t) = i \frac{A e^{+i\frac{\Delta_d}{2}t}}{\Omega_R} \sin\left(\frac{\Omega_R}{2}t\right),$$

$$\Delta_d = \omega_q - \omega_d \text{ and } \Omega_R = \sqrt{A^2 + \Delta_d^2}.$$

$$P_e(t) = |C_e(t)|^2 = \frac{A^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R}{2}t\right).$$



Decoherence of a Qubit

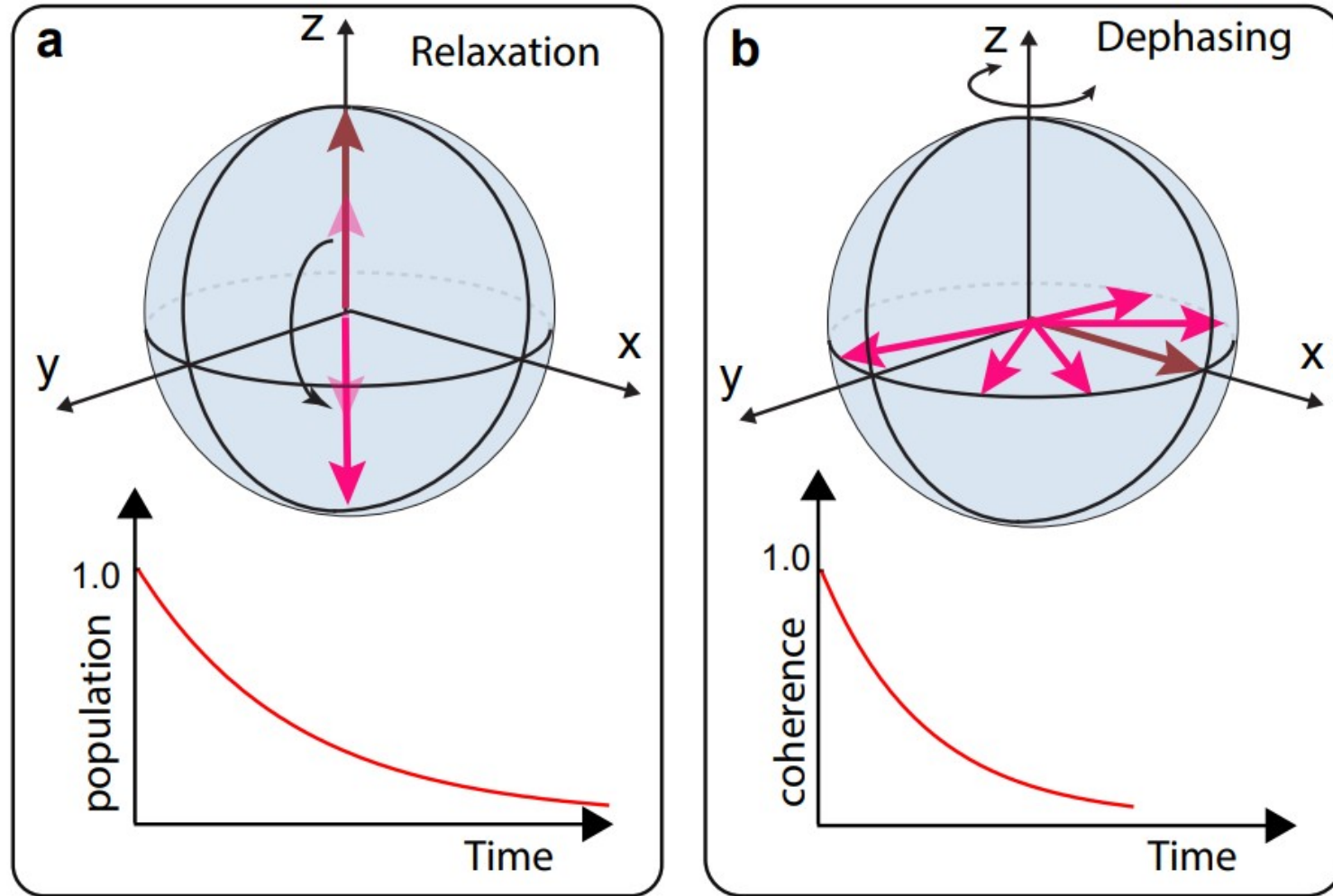


Figure 2.16: Relaxation and dephasing of a qubit.

Simulation of cavity measurements

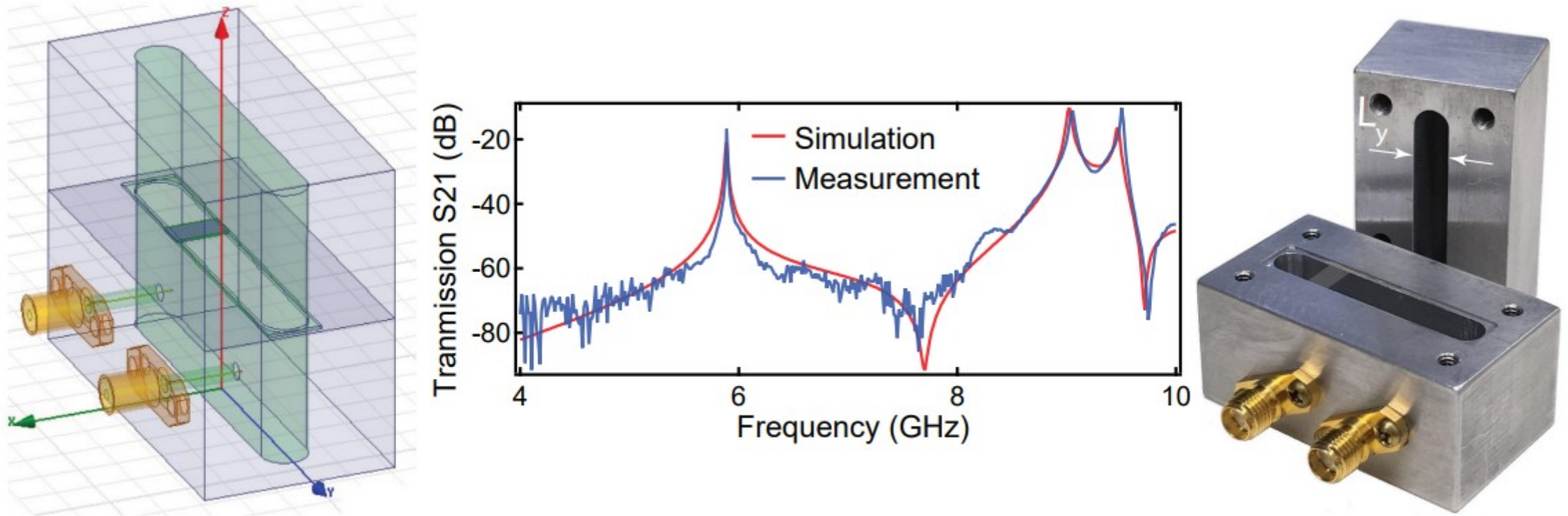


Figure 3.2: **HFSS simulation for cavity transmission:** The cavity transmission is simulated by HFSS (red curve) which is in agreement with actual measurement (blue curve).

The cavity linewidth

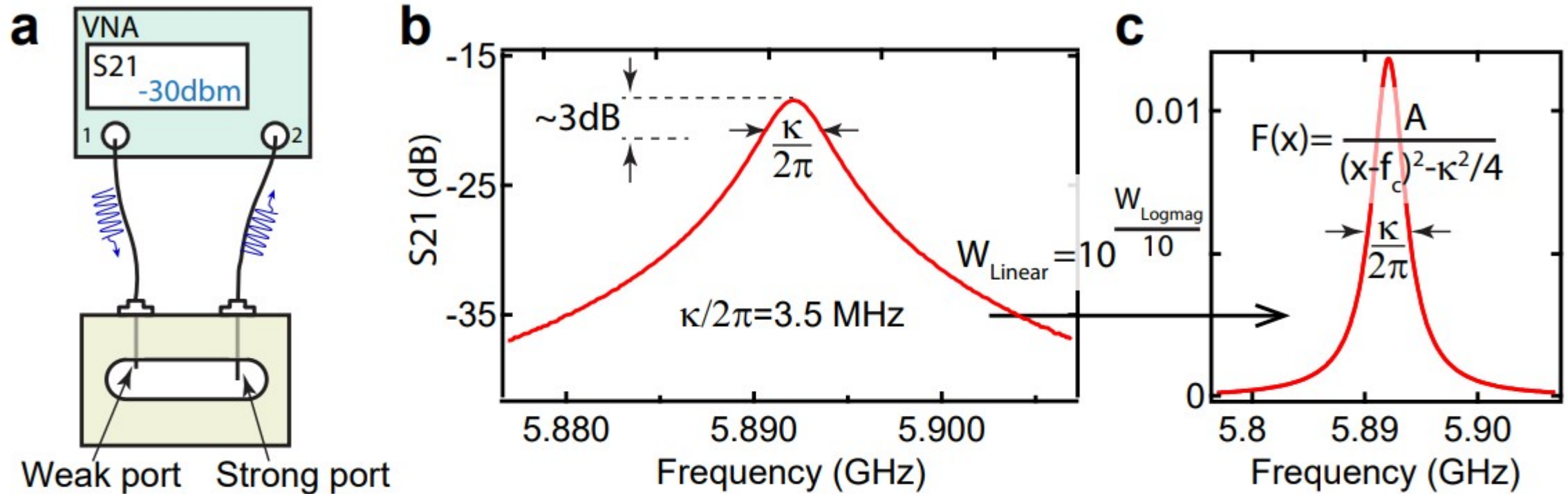


Figure 3.3: **The cavity linewidth characterization:** The cavity linewidth κ can be quantified by measuring the scattering parameters of the cavity. **a,b** In the transmission measurement S_{21} , the cavity linewidth can be estimated by the bandwidth that the transmission signal drops by 3 dB. **c**, More carefully, one can scale the transmission in the linear form and fit to the Lorentzian function. The FWHM of the Lorentzian function would be the cavity linewidth.

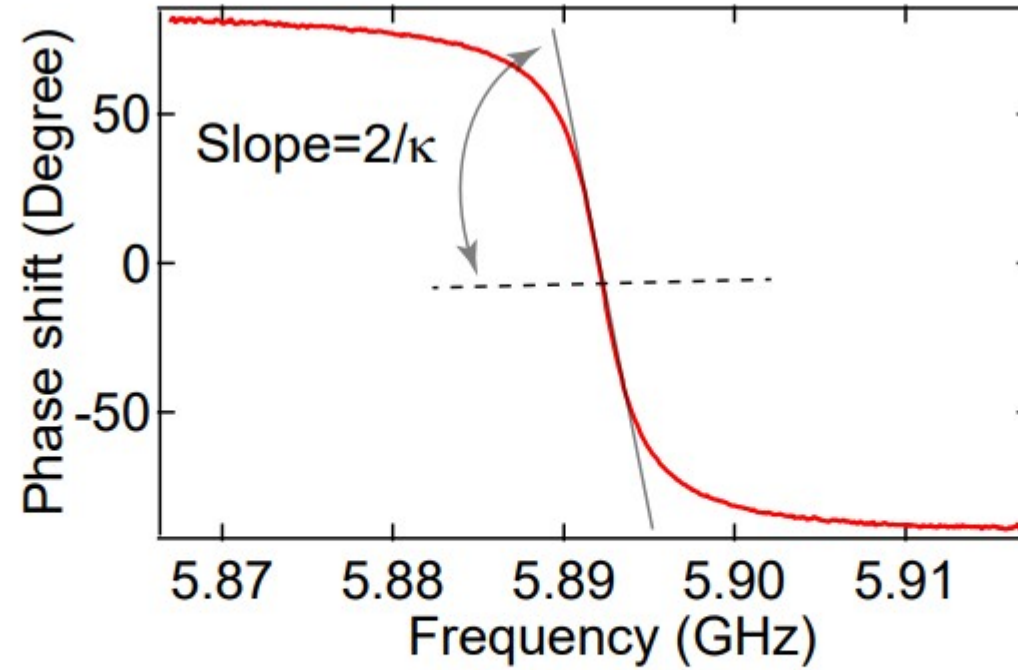


Figure 3.4: **The cavity phase shift across the resonance:** The transmitted signal through the cavity experiences a π phase shift across the resonance of the cavity. Near resonance, the phase shift is approximately linear with a slope of $2/\kappa$.

Cavity/resonator internal and external quality factors

$$\frac{1}{Q_{\text{tot}}} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}}.$$

Transmon Fabrication – NOT included

Qubit-cavity System Characterization

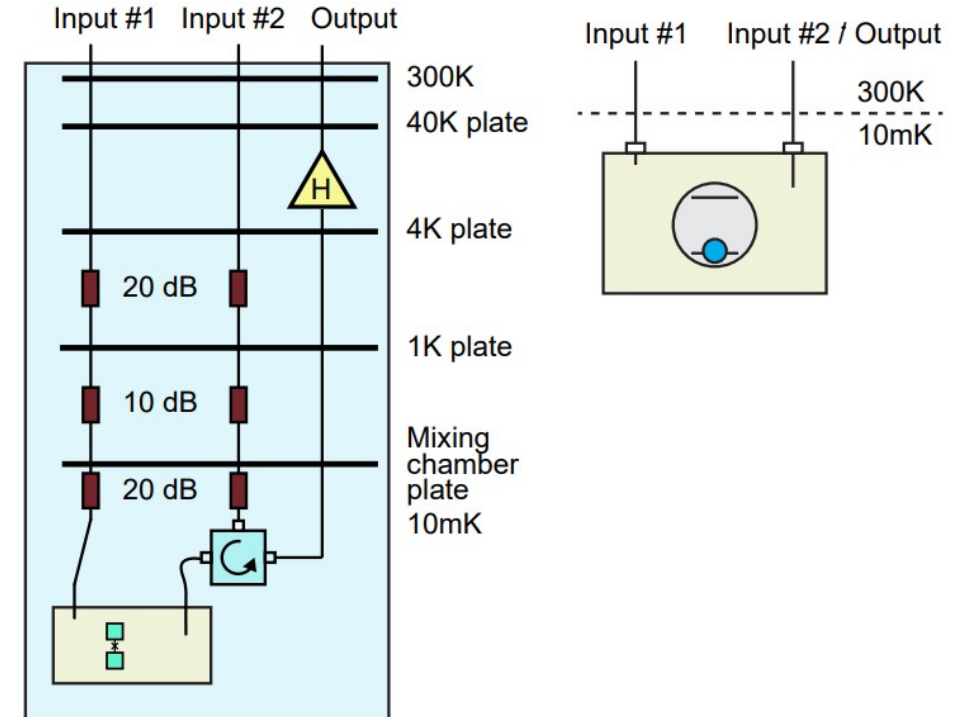
In this section, we discuss a typical qubit characterization procedure. Here are the typical steps before the cool-down:

- Josephson junction room temperature resistance probe,
- Choosing a proper cavity and weak/strong pin length adjustment.
- The qubit placement inside the cavity.
- Characterizing the cavity transmission (qubit chip included).

Qubit-cavity System Characterization

Then we put the cavity-qubit system inside the fridge and cool them down. The minimum circuitry inside the fridge is depicted in Figure 3.11. The main qubit characterization includes five basic experiments.

- One-tone spectroscopy, or “punch-out”.
- Two-tone spectroscopy.
- Rabi measurement.
- T_1 measurement.
- Ramsey measurement (T_2^*).



The first two experiments are in the frequency domain which means we only look at scattering parameters of the system for characterization. However, the **last three experiments are measured in the time domain** and involve preparation and readout of the qubit state. In the following sections, we discuss how these are performed in the lab and what we learn from each experiment in more detail.

One-tone spectroscopy: “punch-out”

The first step is to check if the qubit is “alive” or not. For that, **we need to check whether the cavity frequency shifts based on the state of the qubit**. Of course, at this point, we don’t know the qubit frequency to carefully manipulate it. Fortunately, we don’t need it to know what is the qubit frequency to check if it is there. One way to think about that is if the qubit is coupled to the cavity, the cavity becomes hybridized with the qubit and we should be able to detect a little bit of nonlinearity in the cavity

All we need to do is compare the transmission (or reflection) of the cavity in **low power versus high-power** and see if the frequency of the cavity shifts.

All we need to do is compare the transmission (or reflection) of the cavity in low power versus high-power and see if the frequency of the cavity shifts. When we probe the cavity with very low power we are pretty sure that qubit is in its ground state¹. Therefore we measure the resonance frequency of cavity when qubit is in the ground state. Next, we turn up the power of the VNA to a very high power. In this case, we send a huge amount of photons into the cavity which essentially overwhelms the qubit. Basically, the driving amplitude is so high that the induced current exceeds the critical current of the junction. Practically, in such a high power regime, we measure the bare cavity frequency. Now if there is a working qubit inside the cavity we can see the cavity frequency shift and we say that the cavity “punched out”.

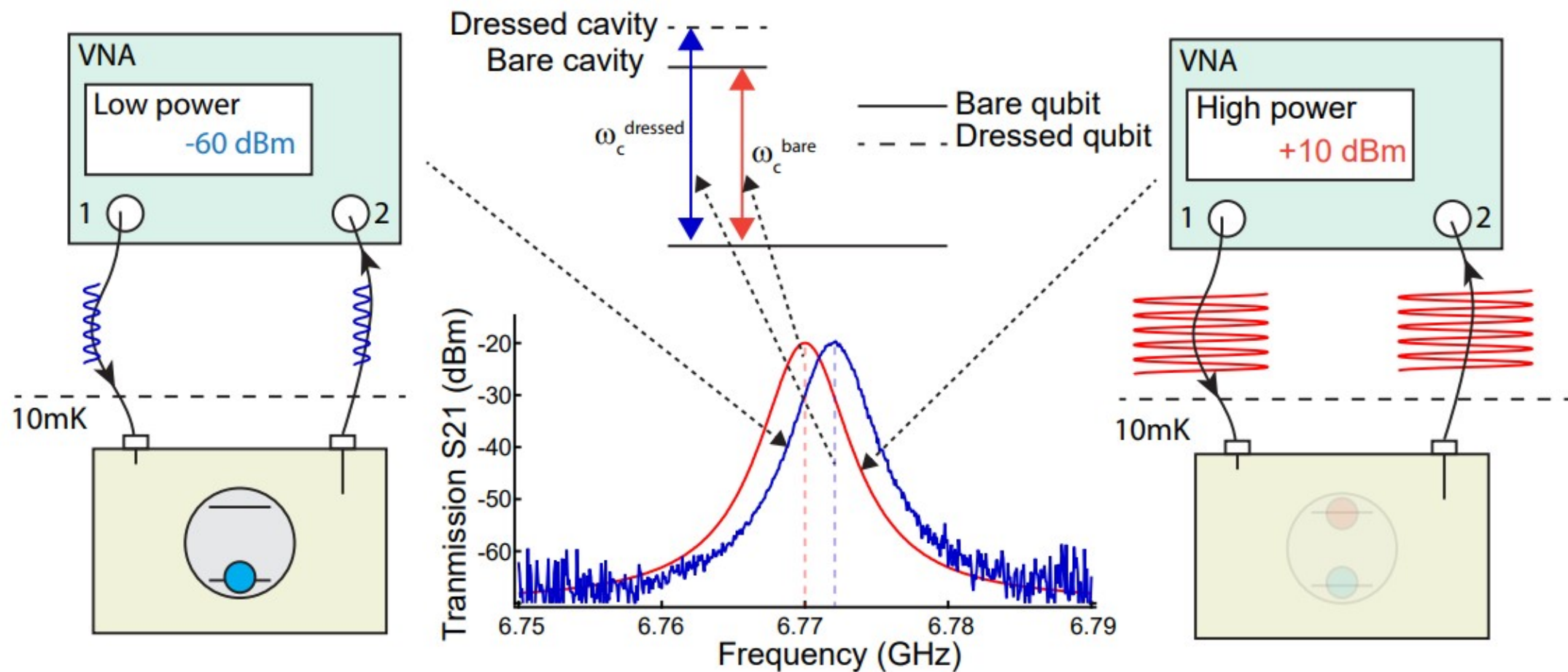


Figure 3.12: **The “punch-out” measurement:** The low (high) power transmission of the cavity indicated by the blue (red) trace. For low power probe signal, the qubit remains in the ground state and we essentially measure the dressed cavity transition indicated by the blue double-sided arrow. In high power case, the qubit is “washed out” and we essentially measure the bare cavity transition as depicted by the red double-sided arrow.

Two-tone spectroscopy

Knowing that the qubit is working, **the next step is to find the qubit frequency**. The idea is to continuously send a **weak microwave signal to the cavity at the low power cavity resonance**—the cavity frequency when the qubit is in the ground state—and probe the cavity transmission. **Therefore, we constantly receive a high transmission signal because we probe at the resonance of the cavity.** While this first tone is on, we start sending another microwave signal (labeled as BNC1) into the cavity.

We sweep the frequency of the probe tone BNC and monitor the cavity transmission as depicted in Figure 3.13. During the sweep, **once the BNC frequency hits the qubit transition frequency** ($\text{BNC} = \omega_q$), it excites the qubit therefore the state of the qubit is no longer in ground state (on average) and that **causes a shift in the cavity frequency**. Now, because the VNA frequency (which is fixed) is no longer resonant with the cavity, the transmitted power drops as depicted in Figure 3.13b.

If we increase the BNC signal amplitude (by ~ 10 dBm), we can also see a second dip at a slightly lower frequency. This dip corresponds to the process by which two photons of drive excited the qubit from the ground state to second excited state, $|g\rangle \rightarrow |f\rangle$ as depicted in Figure 3.13c

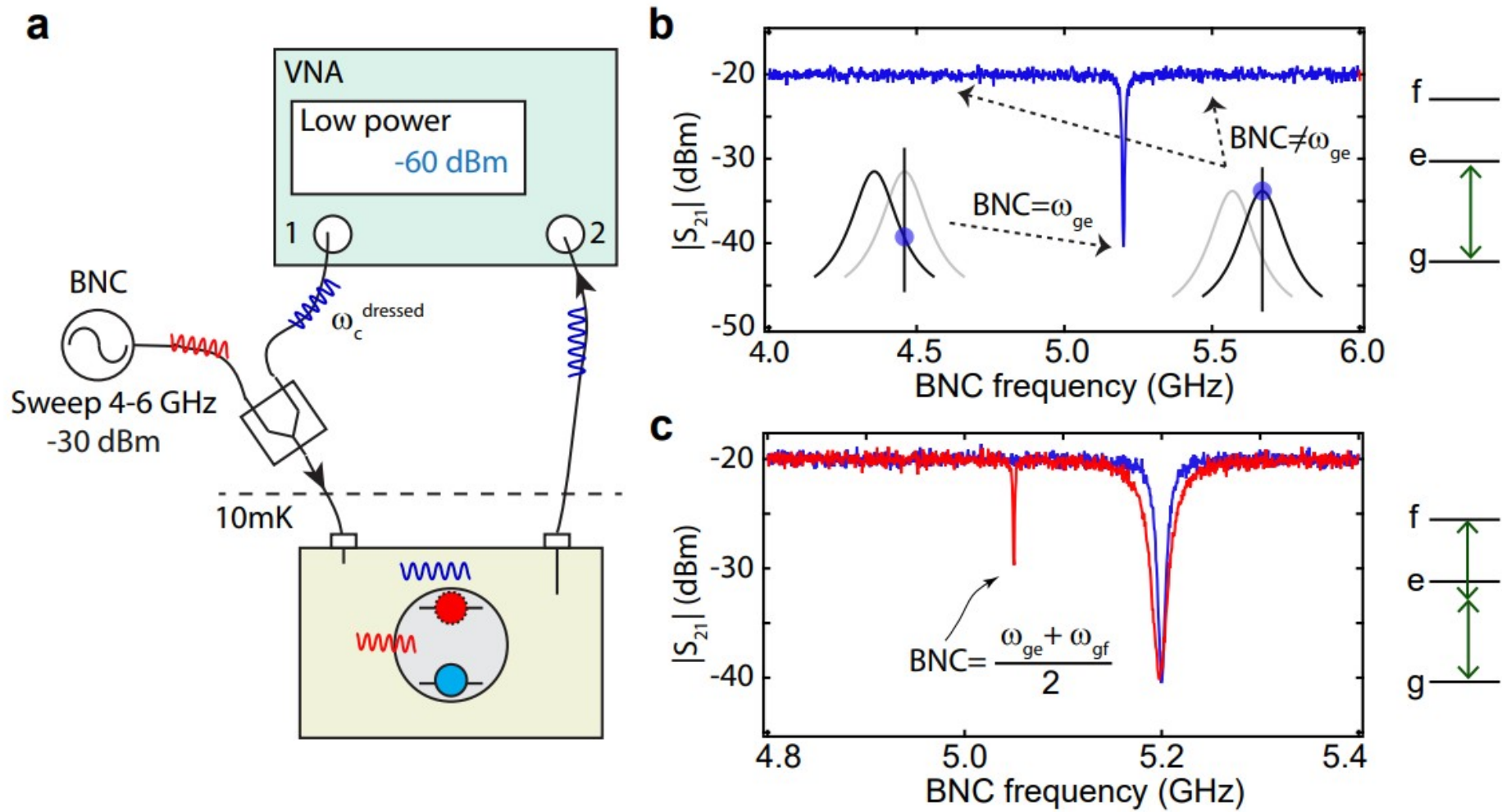


Figure 3.13: **Two-tone spectroscopy:** **a**, The schematics for experimental setup for two-tone spectroscopy. A constant fixed frequency signal from the VNA at the cavity low power probes the cavity while a signal from the BNC sweeps across different frequencies from 4 to 6 GHz. **b**, The cavity transmission versus BNC frequency shows a dip at qubit frequency. **c**, With higher power for BNC the two-photon transition is also detectable (red trace). Note that the qubit transition is power broadened.

Time domain measurement: basics

In this section, we will discuss the time domain measurement of the qubit. **Time domain measurements require initialization, preparation and manipulation of the qubit state, and readout.** In what follows, we briefly discuss these three steps.

Initialization– In our case the initialization is quite simple. The lifetime of the system is on the order of tens of microseconds, therefore all we need to do is leave the qubit for some amount of time (~100 microseconds) to make sure it is in ground state with fairly high fidelity.

Preparation/Manipulation– Unlike the spectroscopy measurements where we constantly send signals and measure the scattering parameters, **in time domain measurements we need to carefully send signals to the system with accurate timing and proper duration.** This means we need to be able to switch the signals on and off with reasonable accuracy.

We use analog RF I/Q mixers to perform switching. Mixers can be used for modulating and demodulating. For switching purposes, we use mixers as a modulator. As depicted in Figure 3.14a, a typical I/Q mixer, ideally, multiplies the LO signal by signals in port I and Q with 90-degree phase difference. Therefore one can use an I/Q mixer in modulation mode to switch a continuous signal. For that, we send the continuous signal to LO port and we switch it by a DC pulse on port I or Q. The RF port output is only a segment of the continuous signal LO, as gated by the DC pulses depicted in Figure 3.14

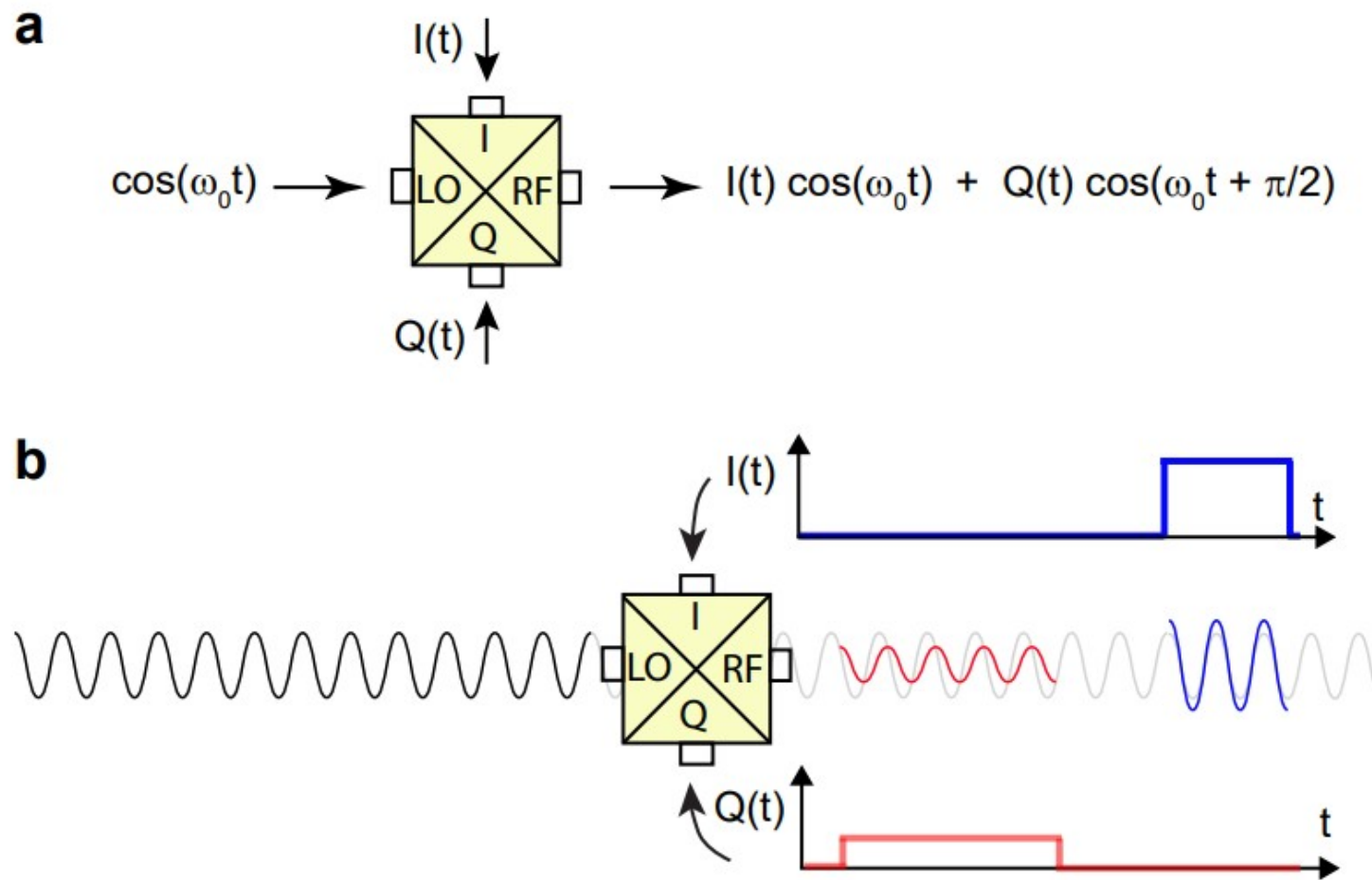


Figure 3.14: *I/Q mixer*: **a**, An *I/Q* mixer can be used as a modulator. Note that the outputs correspond to *I* and *Q* pulses are out of phase by 90 degrees. **b**, The DC pulse in *I/Q* ports can be used to control (switch) a continuous signal. The blue (red) DC pulse is the input for port *I* (port *Q*) and its corresponding output is in-phase (90 out-of-phase) with respect to the local oscillator.

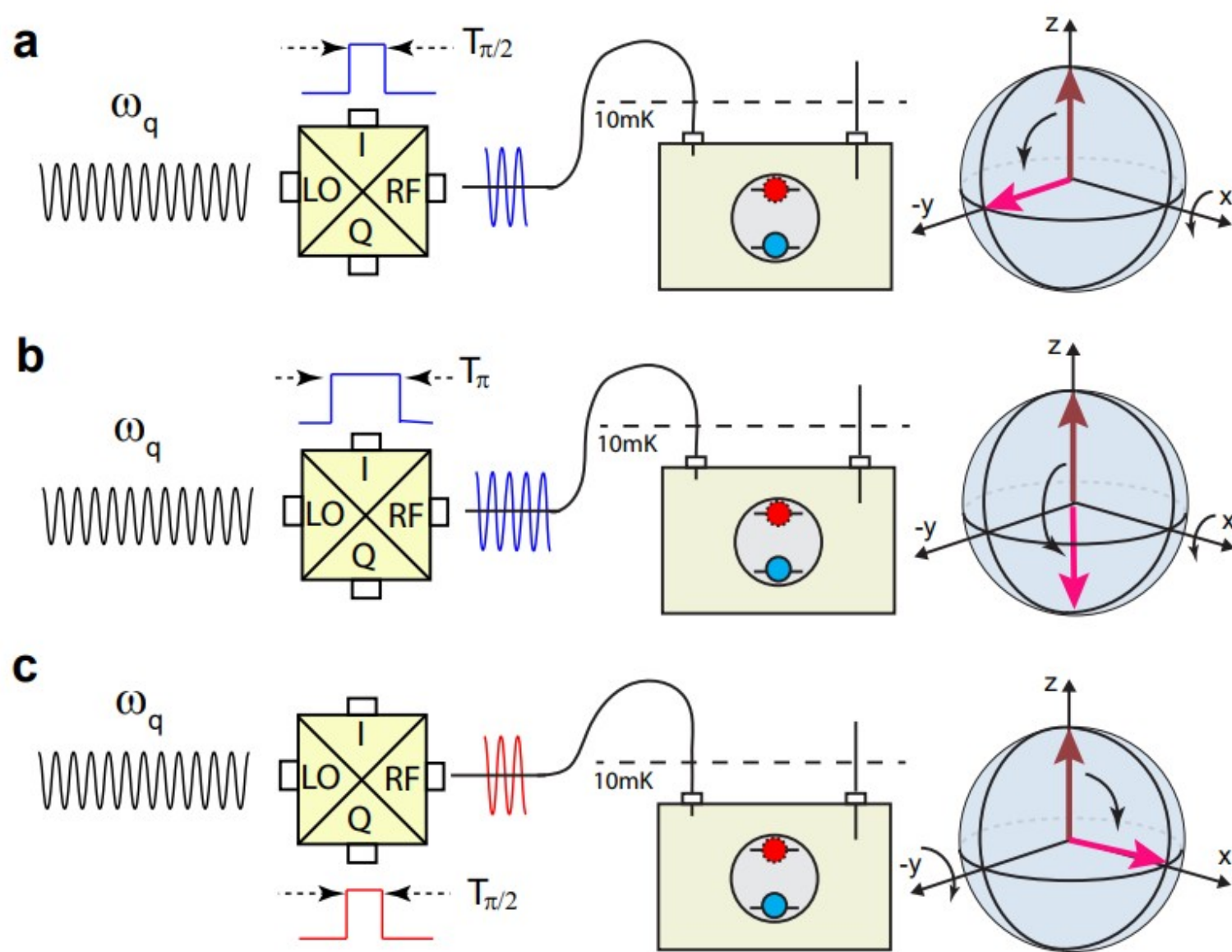


Figure 3.15: **Qubit rotation pulses:** Pulses in the I, Q ports results in rotation in x, y directions in the Bloch sphere. **a**, A $\pi/2$ pulse in port I rotates the qubit along the x direction and prepares a superposition state along y axis. **b**, π pulse prepares the qubit in the excited state. **c**, A $\pi/2$ pulse on Q rotates the qubit around the y axis and prepares the qubit in a superposition along x axis.

Sideband modulation – NOT required

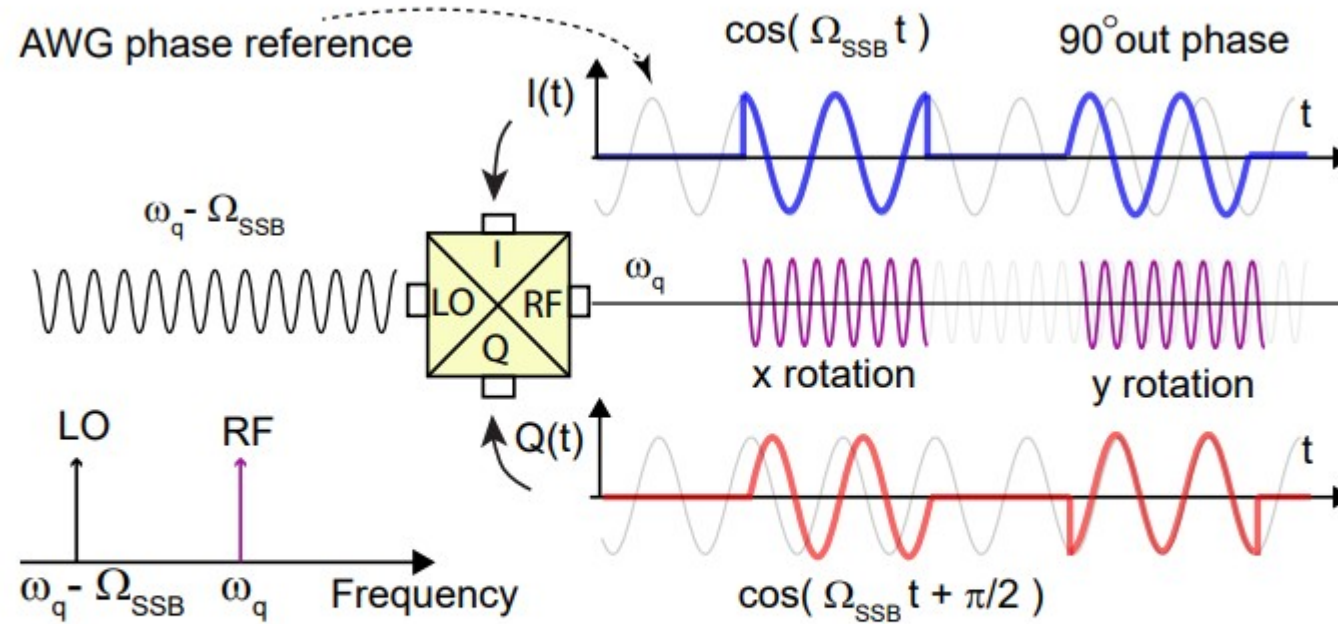


Figure 3.16: **Single Sideband Modulation (SSB)**: By up-converting the LO signal by Ω_{SSB} the mixer outputs at qubit frequency. The relative phase of SSB pulses determines the phase of the output signal thus the direction of the rotation for the qubit.

Readout– We discussed how to manipulate the qubit state by sending signals with frequency ω_q into the system to rotate the qubit. **Here we discuss how to actually measure the qubit state by sending a signal at the cavity frequency ω_c .** As we discussed in the previous chapter, the frequency of the cavity depends on the state of the qubit. Therefore, the natural way to detect the state of the qubit is to measure the phase shift across the cavity by **using homodyne measurement**. We take two copies of a coherent signal that has the frequency of the cavity and **whenever we decide to readout the state of the qubit, we send one to the copies to the cavity and take the transmitted (reflected) signal back and demodulate it with the other copy.** The demodulation results in a DC signal corresponding to the phase difference between two copies. By reading the DC signal we can infer whether the cavity frequency shifted up or down. Figure 3.17 demonstrates the readout process.

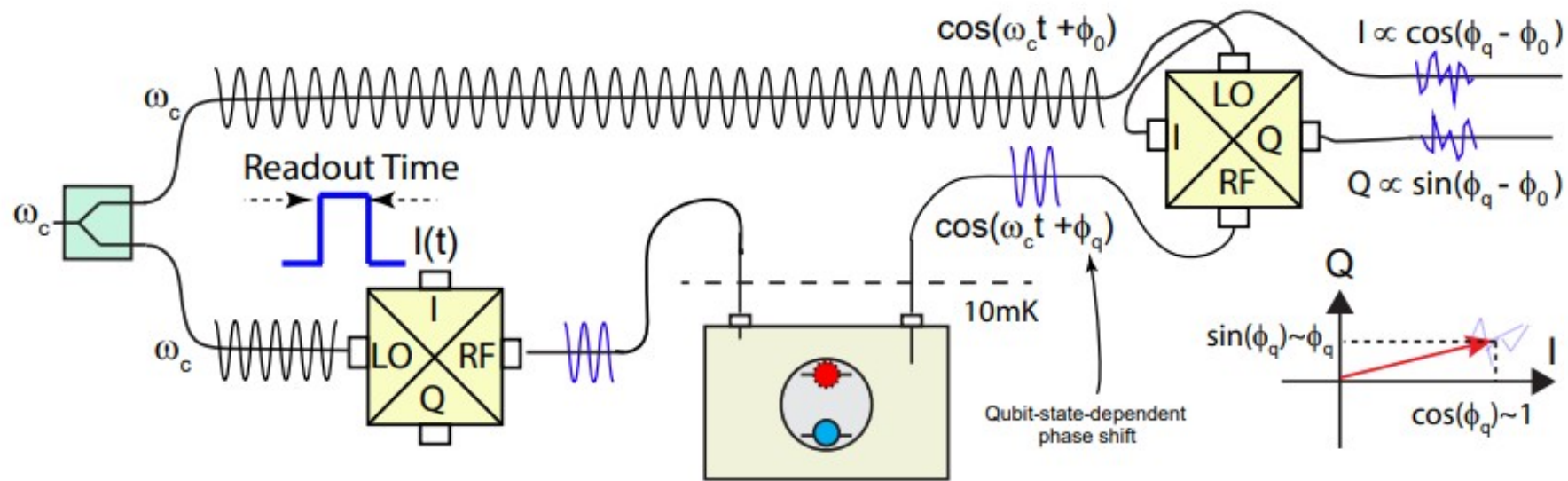


Figure 3.17: **Qubit state readout, homophone detection:** The qubit-state-dependent phase shift is detected by comparing the signal which passes the cavity with the reference signal.

The phase ϕ_0 accounts for the fixed phase difference between two signals due to the different path length and can be set to zero by adding a phase shifter ($\phi_0 = 0$). Therefore the demodulation gives $I = \cos(\phi_q)$ and $Q = \sin(\phi_q)$. For most practical situations the phase shift is small $\cos(\phi_q) \approx 1$, and therefore the phase shift (information about qubit state) is encoded in only in one quadrature of the reflected signal $Q = Q = \sin(\phi_q) = \phi_q$. The entire readout process can be simply represented in a phasor diagram (see Figure 3.18).

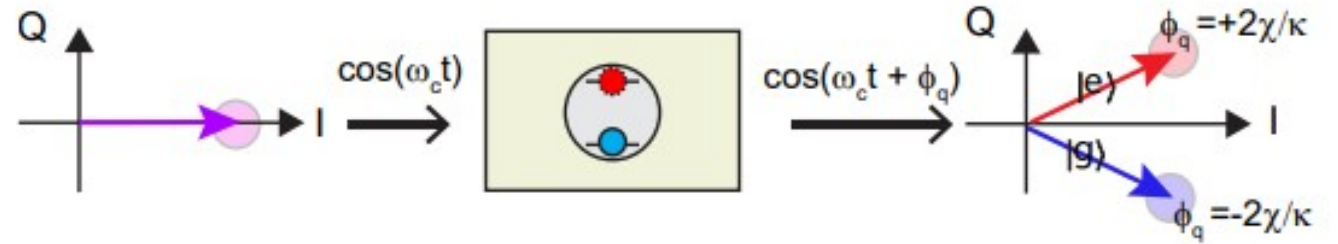


Figure 3.18: **Readout in phase space representation:** A coherent signal with a minimum uncertainty in each quadrature probes the cavity whose frequency shifts by $\pm\chi$ depends on the state of the qubit. The transmitted (reflected) signal acquires a state-dependent-phase shift $\pm 2\chi/\kappa$ as discussed in Section 3.1.

In order to measure the qubit state, we need to repeat the experiment N times ($N > 100$) and each time we detect the phase shift by a value in the Q quadrature. For positive (negative) values we assign -1 ($+1$), indicating that we have found the qubit in the excited (ground) state. After repeating the experiment N times, we gather these statistics and report the population for ground and excited state as

$$P_g = \frac{N_+}{N_+ + N_-} \pm \sqrt{\frac{N_+ N_-}{N^3}} \quad , \quad P_e = 1 - P_g,$$

where $N_{\pm}$ is the number of the experiments where we found the qubit in ground (excited) state inferred by positive (negative) values for Q . The error $\sqrt{N_+ N_- / N^3}$ is calculated based on binomial error.

Rabi measurements

Now we are ready to discuss Rabi measurements. **Once we know the qubit frequency from spectroscopy, then the first experiment in the time domain is to see Rabi oscillation**, since this experiment doesn't require any pulse calibration. **The idea is to send a qubit signal** (with duration t) **to the system and right after that send the cavity readout pulse and readout the state of the qubit**. Normally, we repeat this experiment for different durations t and for each timestep, we repeat the experiment N times. Figure 3.19 summarizes the Rabi experiment setup and procedure.

Normally we sweep t from 0 to 100 ns in 100 steps. In each step the experiment is repeated $N \sim 200$ times to have a reasonably small binomial error so we can see clear Rabi oscillations

The reason we use a rather short Rabi time (100-200 ns) for Rabi measurement is because we will use this Rabi sequence to calibrate preparation pulses (π -pulse and $\pi/2$ -pulse) which are normally short pulses. In this step, we may also tweak the readout power, frequency, and phase to maximize the oscillation contrast as depicted in Figure 3.20a

Rabi measurements

Two pulses: **Red** to change the state of qubit, 2) **Blue** to read the state of qubit

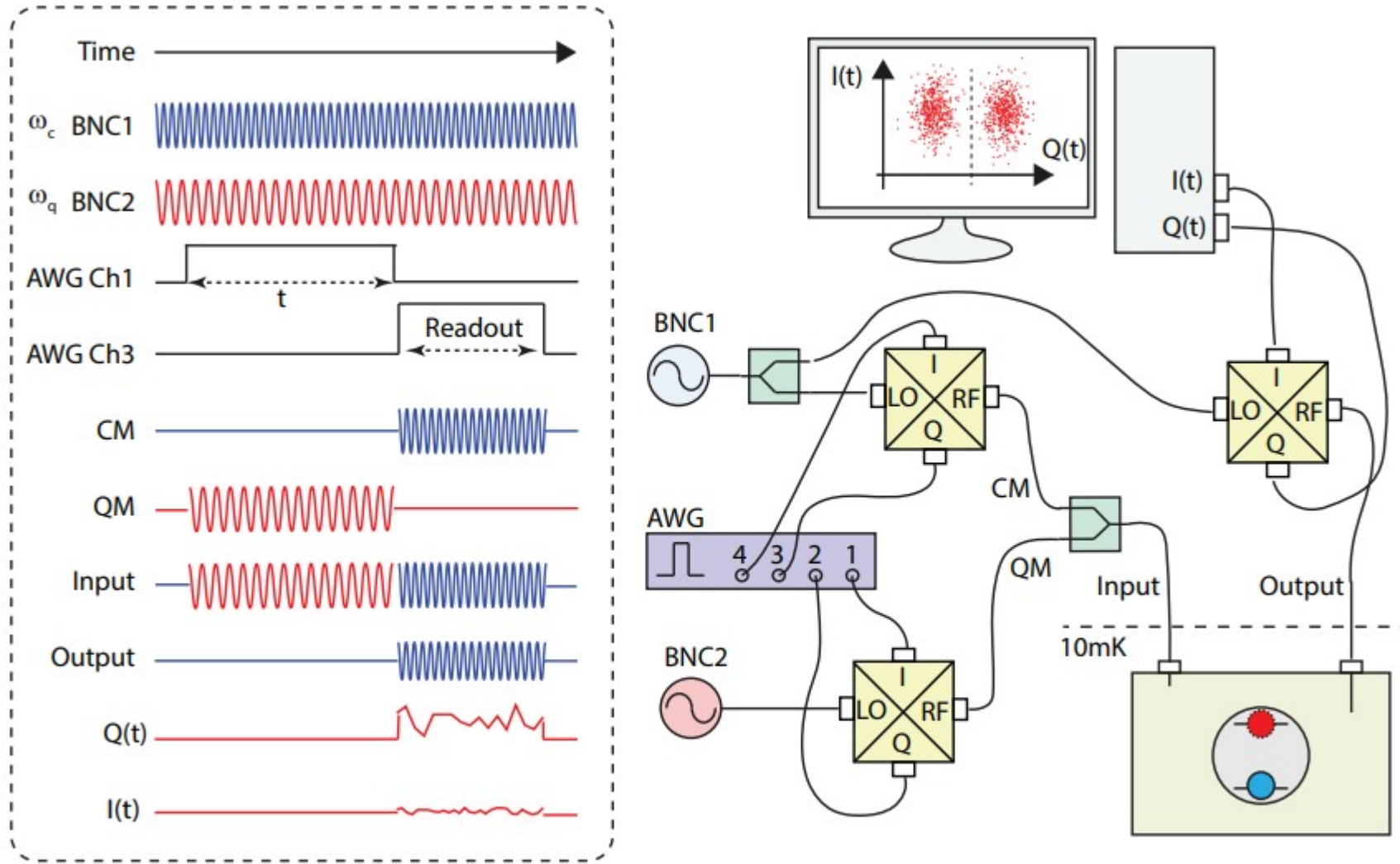


Figure 3.19: **Rabi measurement:** The sequence for the measurement of Rabi oscillations and the typical room temperature circuitry are shown.

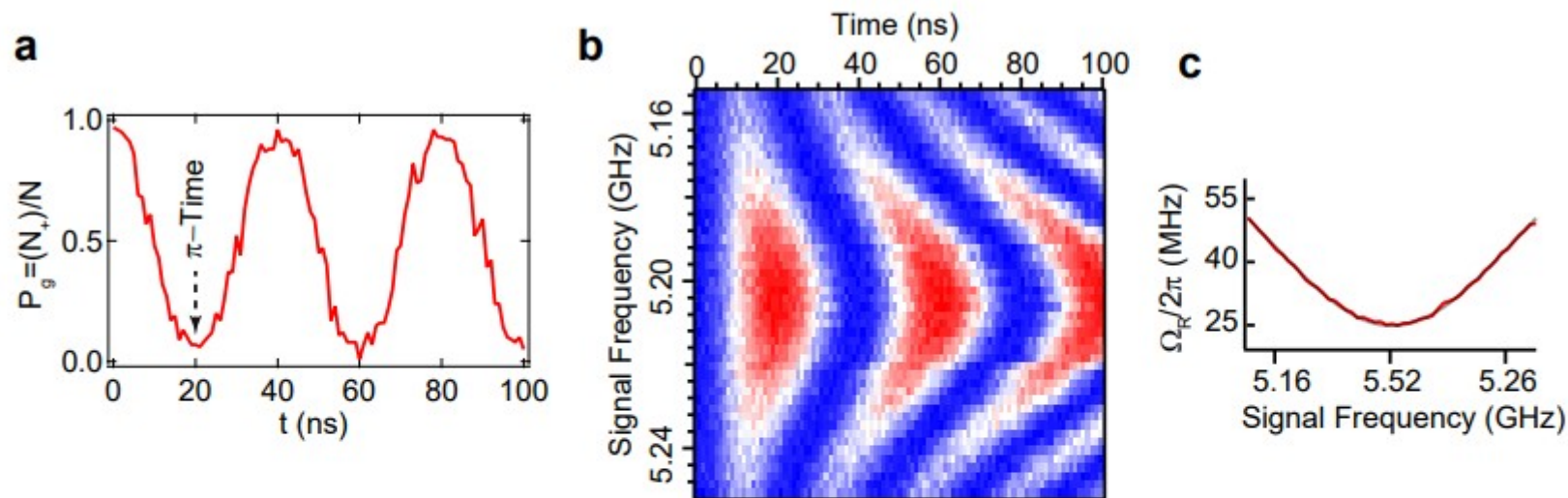


Figure 3.20: **Chevron plot:** **a**, A typical result of Rabi oscillation measurements. **b**, By repeating the Rabi measurement for different qubit frequencies we obtain the “chevron plot” which can be used to calibrate the qubit frequency. **c**, The Rabi oscillation frequency versus qubit pulse frequency. As we discussed in Chapter 2, the minimum oscillation rate corresponds to the maximum contrast for Rabi oscillations which happens when the qubit pulse is on-resonance with qubit frequency.

T_1 Measurement

One of the main characteristics of a qubit is its relaxation time. In order to measure the lifetime of the qubit we do the following sequence:

- 1) **We prepare the qubit in the excited state by sending a π -pulse to the qubit.**
- 2) **We wait for time t , then**
- 3) **we measure the qubit state by sending readout pulse to the cavity.**

Therefore we use the same setup that we had in for Rabi oscillation measurements (see Figure 3.19 for schematics) but we use a slightly different sequence from the AWG. Figure 3.21 summarizes the sequence and a result we get from a T_1 measurement

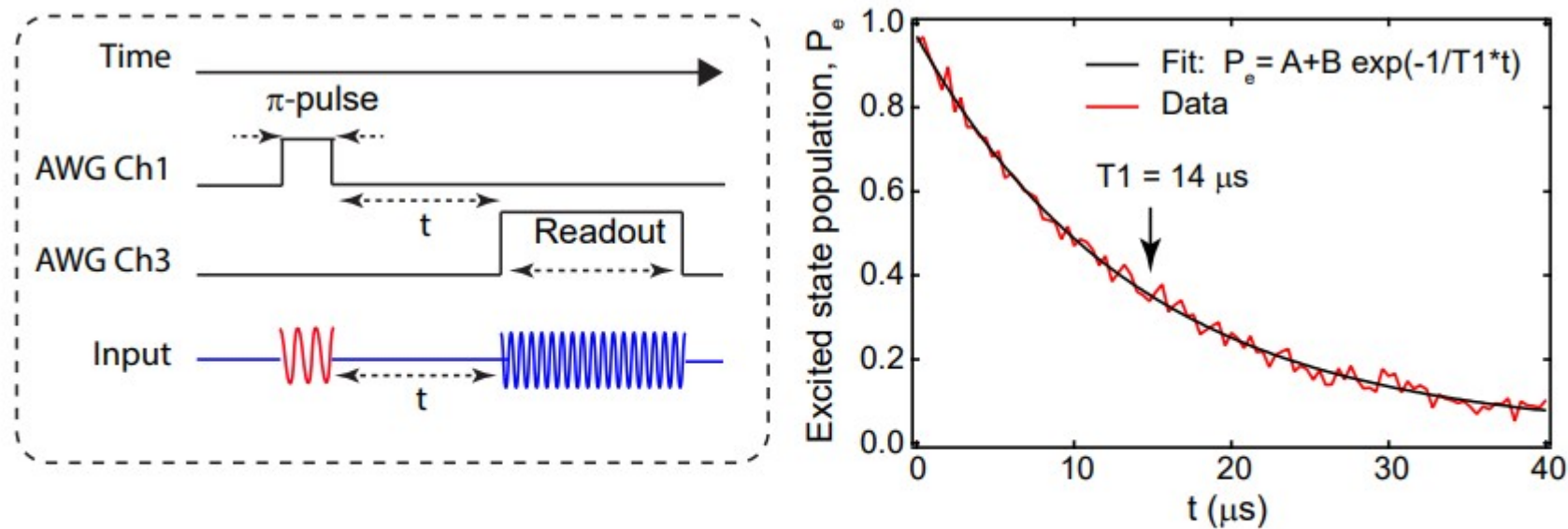


Figure 3.21: T_1 Measurement: The T_1 measurement sequence (consider the experimental setup depicted in Figure 3.19) and a typical result. The qubit lifetime (relaxation time T_1) can be measured by fitting the result to an exponential decay function.

Ramsey Measurement (T_2^*)

With Ramsey measurement, we **characterize the dephasing time for the qubit**, T_2^* . For that, again we use the same setup as we had for measurements of Rabi oscillations (see Figure 3.19). The Ramsey sequence follows as: 1) prepare the qubit in superposition state $\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$ by applying a $\pi/2$ -pulse to the qubit. 2) Wait for a time, then t 3) apply another $\pi/2$ -pulse to bring back the qubit to ground state and immediately 4) perform readout. The sequence for Ramsey measurement and the result has been shown in fig 3.22.

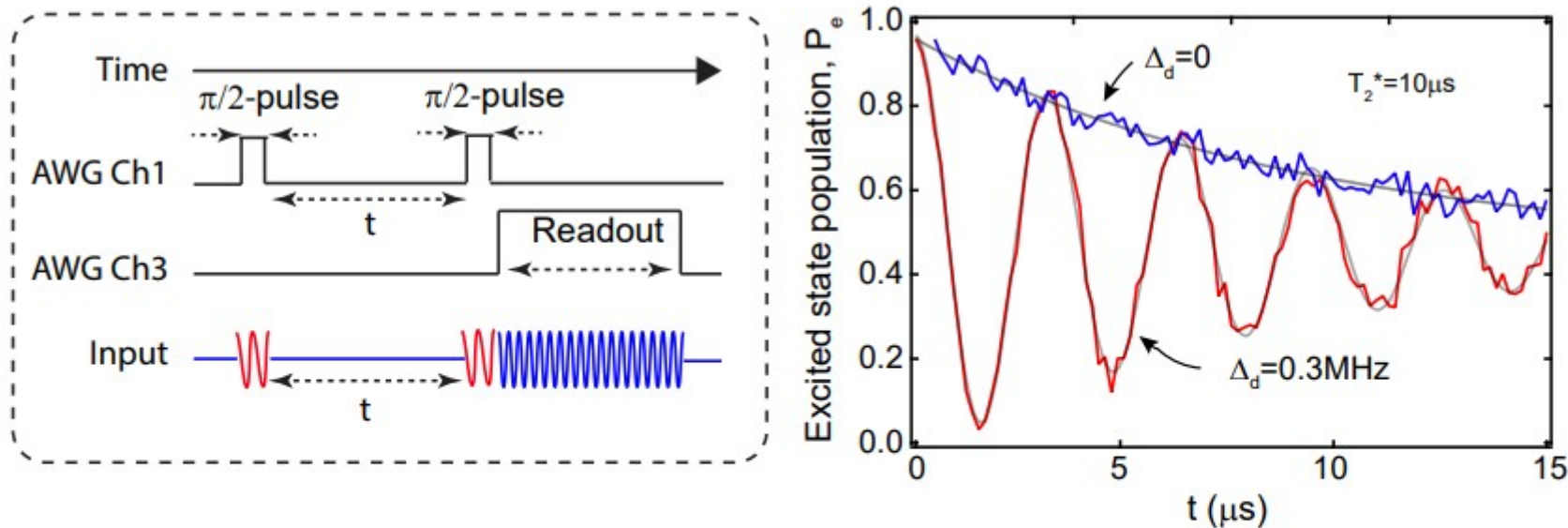


Figure 3.22: **Ramsey measurement:** The sequence for the Ramsey measurement (consider the experimental setup depicted in Figure 3.19) and the typical Ramsey result. The blue (red) curve is when the pulses are on-resonance (0.3 MHz off-resonance) with the qubit frequency. The dephasing time T_2^* can be determined by fitting the data to exponentially decaying sine function $F(t) = A + B \sin(2\pi\Delta_d t + \phi) \exp(t/T_2^*)$.

Full state tomography

The qubit readout projects the state of the qubit along z axis. Therefore one can determine the expectation value for σ_z operator as,

$$z = \langle \sigma_z \rangle = P_g - P_e = \frac{N_+ - N_-}{N_+ + N_-} \pm 2\sqrt{\frac{N_+ N_-}{N^3}}.$$

Similarly, one can determine the expectation value for σ_x and σ_y as well by applying a 90 degree rotation pulse along y and x respectively right before the readout . Therefore, a full tomography sequence has 3 copy of each sequence with no rotation (z), $\pi/2$ -rotation in phase (x), and $\pi/2$ -rotation 90 degree out phase (y) right before the readout as depicted in figure 3.23. The expectation values for σ_x and σ_y , can be determined in a same way,

$$x = \langle \sigma_x \rangle = P_g^{(x)} - P_e^{(x)} = \frac{N_+ - N_-}{N_+ + N_-}$$
$$y = \langle \sigma_y \rangle = P_g^{(y)} - P_e^{(y)} = \frac{N_+ - N_-}{N_+ + N_-},$$

where superscripts (x) and (y) indicate that the readout has in-phase and out-of-phase rotation pulse respectively

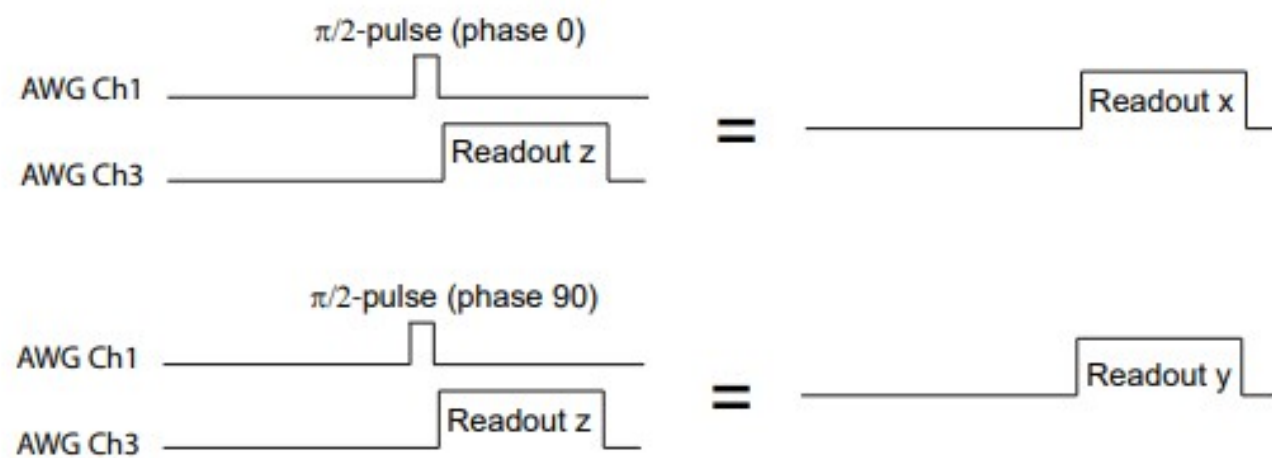


Figure 3.23: **Full state tomography readout pulses:** A readout pulse without any tomographic pulse gives the expectation value for σ_z since it projects the qubit in ground or excited state. A readout pulse augmented by tomographic pulse can be used to measure the expectation values for σ_x, σ_y or any arbitrary basis in the Bloch sphere.

